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1. Introduction

Problem Solving and Decision Making

Problem solving can be defined as a process of recognising a gap between the state of a system as it is and as it is desired to be and taking the necessary measures to make the gap smaller.

Any problem-solving process involves the following steps:

- Identify and define the problem
- Determine the set of alternative actions
- Determine the criteria that will be used for the evaluation of the alternative actions
- Perform the evaluation of the alternatives
- Make the choice
- Implement the chosen alternative
- Evaluate the impact of the chosen alternative to the perceived gap [1].

Decision making is usually associated with the first step of the problem-solving process. Furthermore, problems can be quite complex, thus a decision-maker can require the deployment of a quantitative method to reach a decision. The reasons for using a quantitative method are that quantitative analysis can assist in defining the boundaries of the problem, provide a clear picture of what the purpose is, then provide a clear structure of how it can be solved (and use the same structure to solve similar problems in the future) and finally offer the decision-maker an alternative/solution that can be easily justified.

To achieve this objective, mathematical models are used. Models are abstract representations of reality (or a situation, an object etc). The value of a mathematical model is that it can provide insights into the real world and assist in solving the problem.

The present output will focus on decision aid models and more specifically:

- Linear Programming (LP): A method that deals with a class of problems where the decision-maker needs to optimize a function but is limited by one or more restrictions
- Data Envelopment Analysis (DEA): A method that allows a decision-maker to evaluate the performance of units (factories, vehicles, bureaus etc.)
- Multiple Criteria Decision Aid (MCDA): A class of methods that help a decision-maker to rank a set of alternative actions based on several criteria.

2. Linear Programming

Linear Programming (LP) deals with problems where there is a need to optimally allocate scarce resources in order to perform activities that are competing with each other [2].

The word "program" in Linear Programming is a historical artifact. The initial problem was a resource allocation problem for the Air Force of the United States of America. The decisions about resource allocation were called "Programs" and the term remained ever since [42].

In more detail, in LP the problem is to maximize or minimize a quantity. This is considered the objective of every LP problem. Furthermore, there are restrictions or constraints that limit the degree to which the objective can be achieved. As a result, the decision-making problem that needs to be solved has the following critical elements:

- The variables that represent the competing activities (or services, products etc.) that the decision-maker can act upon. They are called *decision variables.*
- The function that represents the quantity that needs maximization or minimization. It is called *objective function.*
- The restrictions that limit the degree to which the objective can be achieved are called *constraints.*
- Finally, there is the solution of the LP problem, if it does not violate any of the constraints is called *feasible solution.* The set of all feasible solutions is called the *feasible area*of the problem. If a feasible solution also maximizes or minimizes the objective function, it is called *optimal* [3].

2.1 A maximization problem

The owner of a small telecommunications company manufactures two types of mobile phones, phone A and phone B. The company has set the price at such a level that for each phone A that is being sold the profit is 150€, while for phone B is 200€.However, the company cannot manufacture an infinite number of phones. The government with a new law that aims to protect the working force has declared that only 550 working hours will be available to the company per month (based on the number of its employees). To manufacture both phones, 1 working hour for each is required. Furthermore, for the construction of the phones a very rare mineral is required, and the company has only a limited amount of it, 2000 mg. To manufacture phone A, 2 mg of the mineral are required while for phone B 5 mg.

Moreover, the company produces greenhouse emissions when constructing the phones. An analyst has calculated that 1 μg of harmful emissions are emitted for every phone A that is constructed and 3μg for every phone B. According to EU regulations though, the company cannot emit more than 1000 μg of harmful gases per month in order to protect the environment.Finally, the CEO of the

company has decided that this will be the last time that the company produces phone A and they should not produce more than 280 phones A.
The problem is to help management determine how many of each phone they should produce in order to maximize their profits, while respecting all the constraints.The above problem is a typical LP problem. The following paragraphs will present a series of steps on how to solve such problems. The first step is to translate every important piece of information into a mathematical term or function.Thus, the variables that the decision-maker can act upon are the numbers of each phone that will be produced; these are the *decision variables.*

They are denoted as:

$$x_1 = number\ of\ phonesA$$
$$x_2 = number\ of\ phonesB$$

The problem also clearly states that the objective is to maximize profit. For example, if 1 phone A is sold and 0 phones B then the profit will be:

$$P = 150 * 1 + 200 * 0 = 150$$

If 1 phone A and 1 phone B are sold, then the profit will be:

$$P = 150 * 1 + 200 * 1 = 350$$

As a result, the general form of the profit is:

$$P = 150 * x_1 + 200 * x_2$$

This is the *objective function* of the problem. Finally, there are a number of *constraints* in the problem. These are:

<u>Constraint 1</u>: The working hours for constructing the phones must be less than or equal to the number of hours defined by the government. In mathematical terms:

$$1 * x_1 + 1 * x_2 \leq 550$$

<u>Constraint 2</u>: The use of the mineral must be less that the quality that the company possesses. In mathematical terms:

$$2 * x_1 + 5 * x_2 \leq 2000$$

Since it is known that for each phone A 2mg of the mineral are necessary, and 5 mg for each phone B.

<u>Constraint 3</u>: Emissions produced by the company must be lower that the limit imposed by the EU. In mathematical terms:

$$1 * x_1 + 3 * x_2 \leq 1000$$

Constraint 4: The number of phones A must not be more that the limit set by the president of the company. In mathematical terms:

$$x_1 \leq 280$$

Finally, an implicit constraint is that both $x_1 \; and \; x_2$ must be positive; no company can produce a negative amount of something.

As a result, the mathematical model of the LP problem is:

$$MaximizeP = 150 * x_1 + 200 * x_2$$

$$subject \; to \; the \; constraints:$$

$$1 * x_1 + 1 * x_2 \leq 550$$
$$2 * x_1 + 5 * x_2 \leq 2000$$
$$1 * x_1 + 3 * x_2 \leq 1000$$
$$x_1 \leq 280$$
$$x_1, x_2 \geq 0$$

ASSUMPTIONS FOR LINEAR PROGRAMMING

There are 4 assumptions that are necessary in order for a LP model to be appropriate:

- Proportionality: It means that the contribution to the objective function from a decision variable must be linear. The same applies for the constraints. Thus, an objective function of the form $x_2 + x_1^2 + 3$ cannot be part of an LP problem.
- Additivity: there must not be a relationship among the decision variables. Thus, it is not an LP problem if it has the form $x_2 * x_1 + x_1^2$
- Divisibility: All variables must be continuous
- Certainty: All data must be known and precise

The problem that was presented above has only two decision variables, so a graphical procedure can be used to solve it. The first step is to construct a graph with x_1 and x_2 as axes. Then in the graph we design lines that represent the values of x_1 and x_2 permitted by the constraints.

Firstly, x_1 and $x_2 > 0$ so the values lie on the positive side of the axes. Furthermore, the first constraint is:

$$1 * x_1 + 1 * x_2 \leq 550$$

We treat the formula as an equality, and we design the line that corresponds to the equation

$$x_1 = 550 - x_2$$

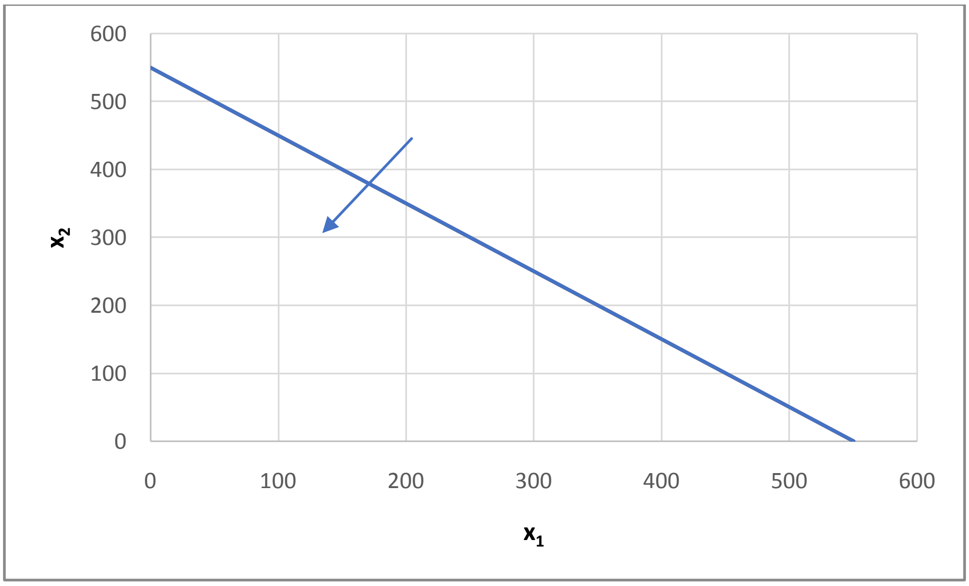


Figure 1.The line that denotes the limit of the first constraint

Next, the area that the inequality applies is shown by the arrow in figure 1. In the figure above (figure 1), it is the triangle that is formed from the horizontal axis, vertical axis and the line. The arrow shows the area. It means that all the values of $x_1$and $x_2$that are inside this area (triangle) solve the first constraint. A similar procedure is followed for all the constraints. The figure below (figure 2) illustrates the graph with all the inequalities that are designed as lines.

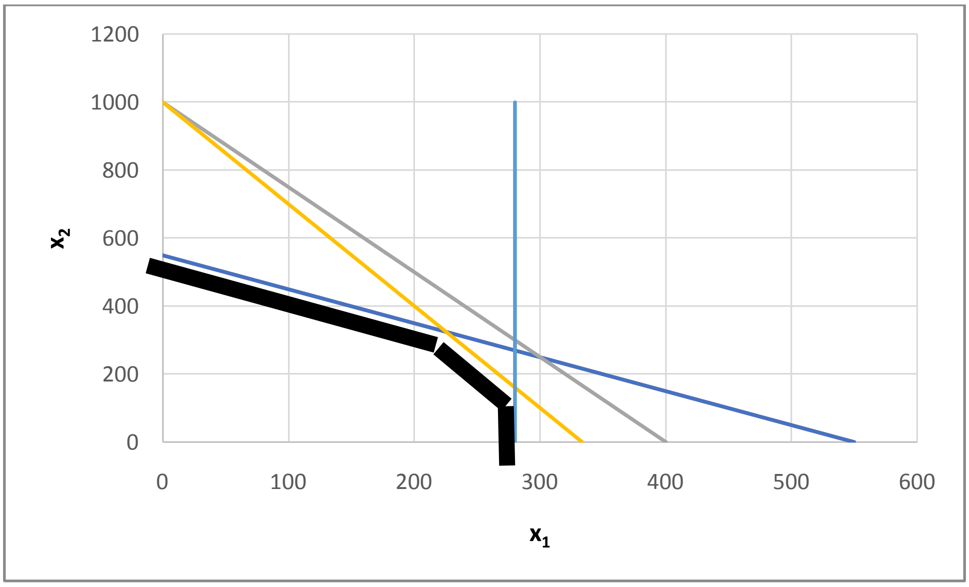


Figure 2. Graph with the feasible area

The common area that is surrounded by the black lines is the *feasible area.* All the values of $x_1$and x_2 that lie within satisfy the constraints. To find the optimal solution, we replace the values of the coordinates of $x_1$and $x_2$that correspond to the *extreme points* (where the lines that surround the feasible area correspond) in the objective function and we choose the pair that maximizes it.

Table 1. Solutions of the extreme points

X_1	X_2	P
0	0	0
0	550	110000
225	325	98750
280	160	74000
280	0	42000

As a result, the optimal solution is to produce 0 of phone A and 550 of phones B. It is important to note that the optimal solution is always one of the extreme points of the feasible area.

It should also be stated that the second constraint,

$$2 * x_1 + 5 * x_2 \leq 2000$$

does not participate in the definition of the feasible area. This means that even if the constraint was not part of the problem, the solution would be the same. Such a constraint is called *redundant.* The second constraint concerned the use of the mineral that was necessary for the manufacturing of the phones, and its non-participation in the solution, means that the company has more than enough for its production schedule.

Questions

- Knowing the optimal values for$x_1$and x_2, what happens to the other constraints? Are all the resources used (time, the limit for emissions)?
- Can you provide an interpretation of your answer to question a?

Question 2:

- Visit the website http://www.phpsimplex.com/simplex/simplex.htm?l=en
- Define a problem with 2 decision variables and 4 constraints.
- Place the values in the cells
- Solve the problem and explain the results. Is the solution to your problem feasible?
- Having found a feasible solution to the maximization problem, find an equivalent problem that fits the values and constraints of the problem.

2.2 A minimization problem

The problem described in the previous section was a maximization problem; the objective function requires such value as to reach its maximum. However, there are LP problems where the aim is to minimize the objective function.

A minimization problem takes the form:

$$\mathit{Minimize} \qquad C_1 * x_1 + C_2 * x_2 + \cdots + C_n * x_n$$

$$\mathit{subject\ to\ the\ constraints:}$$

$$\begin{gathered} a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n \geq y_1 \\ a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n \geq y_2 \\ a_{31} * x_1 + a_{32} * x_2 + \cdots + a_{3n} * x_n \geq y_1 \\ x_1, x_2, \ldots x_n \geq 0 \end{gathered}$$

2.3 Exercise 1

A special type of Linear Programming models is called transportation problems. Their objective is to minimize the transportation cost for moving a product (or people etc) from one point to another, while being subjected to demand and supply constraints. The following paragraphs present such an example (adapted from [4]).

A company needs to minimize the costs of transporting its goods from a set of three factories (Factory 1, Factory 2, Factory 3) to its four stores (Store 1, Store 2, Store 3, Store 4) where it sells the products that are manufactured in the factories.The company knows that the three factories can produce 400, 1500 and 900 units of product at each factory respectively. Moreover, the four stores require 700, 600, 1000 and 500 units of product respectively. However, there is a cost associated to the transportation of the products from each factory to each store. The table (table 2) below presents how much it costs the company to send 1 unit of product from each factory to each store.

Table 2. Transportation costs from the factories to the stores

	Store 1	Store 2	Store 3	Store 4
Factory 1	20	40	70	50
Factory 2	100	60	90	80
Factory 3	10	110	30	200

For example, it will cost the company 20 (monetary/cost units) to send one unit of product from Factory 1 to Store 1, 40 (monetary/cost units) to send one unit of product from Factory 1 to Store 2 etc. In combination with the information on how much each factory can produce (supply) and how much each store wants (demand), the following table (table 3) can be formulated.

Table 3. Information table

	Store 1	Store 2	Store 3	Store 4	*Supply*
Factory 1	20	40	70	50	*400*
Factory 2	100	60	90	80	*1500*
Factory 3	10	110	30	200	*900*
Demand	*700*	*600*	*1000*	*500*	

Based on the information in the final table, the company asks you the optimal way to transport the products from the factories to the stores in order to minimize the costs (problem adapted from [4])

2.4 Exercise 2

Assume that the decision variables are denoted as x_{ij}, where i the factory number (i=1, 2, 3) and j the store number (j=1,2,3,4). For example, x_{12}represents the number of units of products that will be transported from Factory 1 to Store 2 or x_{34} is the number of units of products that will be transported from Factory 3 to Store 4, etc.

- Write the mathematical form of the LP minimization problem described above
- Which of the constraints are about the supply restrictions and which are about the demand restrictions?

The particular problem has 12 decision variables; thus, a graphical solution cannot be used to solve it. However, there are tools that can be used to assist in solving all types of LP problems. One of these tools is Pulp. Pulp [5] is a Python module/package. The code below illustrates how the above problem can be solved in python along with the output/solution of the program.

```
# -*- coding: utf-8 -*-
"""
Transportation Problem
SUSTAIN project
"""

# Import the Pulp module
from pulp import *

# Data for the LP problem
F = 3# The number of factories
S = 4# The number of stores
a = range(1, F+1)
a1 = range(F)
b = range(1, S+1)
b1 = range(S)

#Index list for the decision variables
xindx = [(a[i], b[j]) for j in b1 for I in a1]

#Creation of the model that will contain the data and solve the LP
model = LpProblem("Transportation LP Problem", LpMinimize)

#Creation of the Decision variables
x = LpVariable.dicts("X", xindx, 0, None)

#The problem's objective function
model += 20.0*x[1,1] + 40.0*x[1,2] + 70.0*x[1,3] +50.0*x[1,4]\
+100.0*x[2,1] + 60.0*x[2,2] + 90.0*x[2,3] + 80.0*x[2,4] \
+ 10.0*x[3,1] +110.0*x[3,2] + 30.0*x[3,3] + 200.0*x[3,4] "Transportation cost"

#Constraints that concern the supply
```

```
model += x[1,1] + x[1,2] +x[1,3] +x[1,4] <= 400.0,"Supply of Factory 1"
model += x[2,1] + x[2,2] +x[2,3] +x[2,4] <=1500.0,"Supply of Factory 2"
model += x[3,1] + x[3,2] +x[3,3] +x[3,4] <=900.0, "Supply of Factory 3"
#Constraints that concern the demand
model += x[1,1] +x[2,1] + x[3,1] >= 700.0, "Demand of Store 1"
model += x[1,2] +x[2,2] + x[3,2] >= 600.0, "Demand of Store 2"
model += x[1,3] +x[2,3] + x[3,3] >= 1000.0, "Demand of Store 3"
model += x[1,4] +x[2,4] + x[3,4] >= 500.0, "Demand of Store 4"

#Solve the problem
model.solve()

#Print the solution
print("Status:", LpStatus[model.status])

#Print each of the variables with the values of the optimal solution
for v inmodel.variables():
print(v.name, "=", v.varValue)

#Print the optimal value of the objective function
print("objective function", value(model.objective))
```

The result of the above code is:

```
Status: Optimal
X [1,1] = 400.0
X [1,2] = 0.0
X [1,3] = 0.0
X [1,4] = 0.0
X [2,1] = 0.0
X [2,2] = 600.0
X [2,3] = 400.0
X [2,4] = 500.0
X [3,1] = 300.0
X [3,2] = 0.0
X [3,3] = 600.0
X [3,4] = 0.0
Objective function = 141000.0
```

- Interpret the output of the code

2.5 Exercise 3

In board games there might be a time where it is necessary to place some resources (workers, pawns, buildings etc.) in places on the board. However, each *worker placement* comes with a cost and you only have a specific number of workers (resources) at your disposal. For example, let's assume that you are the mayor of a city and you need to decide on the construction of 3 new bus terminals (named B1, B2, B3). There are four candidate sites S1, S2, S3, S4. However, there is also an expected cost for the placement of a terminal to any of the spots. Table 4 below presents these costs.

Table 4. Cost table for the exercise

	S1	S2	S3	S4
B1	13	16	12	11
B2	15	2	13	20
B3	5	7	10	6

> In the context of board games, cost can be considered as for example the number of rounds that need to pass before the change takes effect, or the number of game coins that need to be spent in order for a movement to occur

However, in this problem each terminal can only be assigned to one site and each site can only receive/accommodate only one terminal.

- What are the decision variables?
- What is the mathematical formulation of the problem?
- Solve the problem in Pulp
- What is the difference between the particular problem and the ones that have been described before?

```
# -*- coding: utf-8 -*-
"""
Assignment Problem
SUSTAIN Project
"""

#Import the Pulp module
from pulp import *

## Data for the LP problem
S = 4# Number of available sites
B = 4# Number of Bus terminals- It includes the dummy variable
NAV = B + S # Number of artificial variables
a = range(1, S+1)
a1 = range(S)
b = range(1, B+1)
b1 = range(B)

#Index list for the decision variables
xindx = [(a[i], b[j]) for j in b1 for I in a1]
tindx = range(1, NAV+1)

#Creation of the model that will contain the data and solve the assignment LP
model = LpProblem("Assignment Problem", LpMinimize)

#Creation of the Decision variables
x = LpVariable.dicts("X", xindx, 0, None)
y = LpVariable.dicts("Y", tindx, 0, None)
```

```
#The problem's objective function
model += 13.0*x[1,1] + 16.0*x[1,2] + 12.0*x[1,3] +11.0*x[1,4] +\
15.0*x[2,1] + 2.0*x[2,2] + 13.0*x[2,3] + 20.0*x[2,4] \
+ 5.0*x[3,1] +7.0*x[3,2] + 10.0*x[3,3] + 6.0*x[3,4], "Assignment cost"

#Constraints that concern the sites
model += x[1,1] +x[2,1] + x[3,1] + x[4,1] - y[1] >= 1, "Site 1"
model += x[1,2] +x[2,2] + x[3,2] + x[4,2] - y[2] >= 1, "Site 2"
model += x[1,3] +x[2,3] + x[3,3] + x[4,3] - y[3] >= 1, "Site 3"
model += x[1,4] +x[2,4] + x[3,4] + x[4,4] - y[4] >= 1, "Site 4"

#Constraints that concern the terminals
model += x[1,1] + x[1,2] +x[1,3] +x[1,4] -y[5] >= 1, "Terminal 1"
model += x[2,1] + x[2,2] +x[2,3] +x[2,4] -y[6] >= 1, "Terminal 2"
model += x[3,1] + x[3,2] +x[3,3] +x[3,4] -y[7] >= 1, "Terminal 3"
model += x[4,1] + x[4,2] +x[4,3] +x[4,4] -y[8] >= 1, "Dummy Terminal 4"

#Solve the problem
model.solve()

#Print the solution
print("Status:", LpStatus[model.status])

#Print each of the variables with the values of the optimal solution
for v inmodel.variables():
print(v.name, "=", v.varValue)

#Print the optimal value of the objective function
print("objective function", value(model.objective))
```

The result of the above code is:

```
Status: Optimal
X [1,1] = 0.0          Y1 = 0.0
X [1,2] = 0.0          Y2 = 0.0
X [1,3] = 0.0          Y3 = 0.0
X [1,4] = 1.0          Y4 = 0.0
X [2,1] = 0.0          Y5 = 0.0
X [2,2] = 1.0          Y6 = 0.0
X [2,3] = 0.0          Y7 = 0.0
X [2,4] = 0.0          Y8 = 0.0
X [3,1] = 1.0
X [3,2] = 0.0
X [3,3] = 0.0
X [3,4] = 0.0
X [4,1] = 0.0     Objective function = 18.0
X [4,2] = 0.0
X [4,3] = 1.0
X [4,4] = 0.0
```

- Interpret the output of the code

3. Data Envelopment Analysis

Data Envelopment Analysis (DEA) is a non-parametric[1], mathematical programming technique that is used for the evaluation of the technical efficiency of Decision Making Units (DMUs)[2] relative to one another. DEA's foundations lie in the works of [6]; [7]; [8] and originated in the seminal papers of [9]; [10].Its original inception was to be used as a general-purpose evaluation method for a firm's efficiency, when multiple criteria are involved.

The definition of a DMU's efficiency originates in the engineering disciplines and is defined as the ratio of the sum of its weighted outputs divided by the sum of its weighted inputs [11].

$$technical\ efficiency = \frac{\sum w_{output}\ * y}{\sum w_{input}\ * x}, wherex = inputl\ evel\ and\ y = output\ level$$

In other words, technical efficiency is a measure of how well a DMU can transform its inputs into outputs. For example, in the context of urban metabolism, technical efficiency can be considered how well a city (as a DMU) can transform material and energy (inputs that are necessary for everything to function) to quality services to its citizens (outputs). In the context of transportation, the technical efficiency of a public transportation organization can be considered how well the vehicles and drivers it employs (inputs) are translated into passengers serviced by the particular public transportation mode.

The definition of technical efficiency allows the comparative assessment of DMUs in situations where price information is not available; where for example we could compare how much it costs to manufacture something versus how much are the revenues from its sale. As a result, DEA is considered a suitable method for aggregating economic, social and environmental factors with different units with the purpose of constructing a multi-dimensional indicator of efficiency. This indicator can help a decision-maker to comprehend better which DMUs under his control are efficient, where there is room for improvement and how this improvement can be achieved [11].

Furthermore, DEA has several advantages over other parametric methods of performance assessment:

- It does not require the identification of the relationship between inputs and outputs; we are not concerned what is the process that transforms inputs to outputs. In the above example, we are only studying how well material and energy are transformed into services, but we do not examine what is relationship between those two (how energy is transformed to fuel for a

[1] By non-parametric it is meant that there are no assumptions as to the forms of the parameters, the form of the frequency distributions etc.

[2] DMUs are entities such as people, firms, organizations (public and private) that convert multiple inputs to multiple outputs. In DEA the DMUs are considered homogeneous in terms of identical inputs and outputs

car, that transports people, to go to their job, to gain a salary and thus make a living and have a good quality of life).

- It requires less information than traditional methods
- It can indicate the reasons for which a unit is inefficient and propose plans to improve it [12]; [13].

3.1 Formulation of a DEA problem

A DEA problem is defined as follows:

Suppose there are N homogeneous DMUs, where each DMU uses m inputs to produce s outputs. This is the typical formulation and what the DEA methods tries to achieve is identify the most efficient DMU by maximizing each one's individual efficiency. To do so, each DMU's efficiency is calculated relative to an efficient frontier. DMUs located at the efficiency frontier has an efficiency score of 1 (or 100%), while those under the frontier have an efficiency of less than 1 (or 100%), hence they have the capacity to improve [11].

A history of DEA by its founder can be found:
https://www.youtube.com/watch?v=fxAASxxCeFc

And a summary of the basic notions can be found in:
https://youtu.be/qE2mzqKTI58

3.2 Basic DEA models

There are two basic models of DEA. The first one was proposed by [9] and it assumes that we have Constant Returns to Scale (CRS). This model is appropriate when all DMUs are operating at an optimal size, which further assumes that they are operating in a perfectly competitive environment. These assumptions are considered rather ambitious and for that reason Banker et al [10] extended the original model to one which assumes Variable Returns to Scale (VRS). This model is appropriate when all DMUs are not operating at an optimal size, usually because of imperfect competition, government regulations etc.

Returns to scale are terms that describe what happens in the production when all inputs change by a factor of a, where a>0. There are three interrelated and sequential laws of return to scale:

- **Constant Returns to Scale**: When output increases by the same proportional change as all inputs change.
- **Increasing returns to scale**: In this situation, the average inputs consumption decreases while output rises. For example, a change in the output of 10% results in change in inputs of less than 10%. For a DMU, operating in increasing returns to scale and wish to improve, it means an expansion of output is necessary.
- **Decreasing returns to scale**: In this situation, the average inputs consumption increases while output rises. For example, a change in output of 10% results in change in inputs of more than 10%. When a DMU is operating under decreasing returns to scale and wish to improve, it means a reduction of output.

For a more mathematical definition of returns to scale in DEA, let's assume that a DMU is using x inputs to produce y outputs. If the inputs are scaled by a: a*x, a>0, then the outputs could vary by b: b*y. Furthermore, let's assume:

$$p = \lim_{a \to \infty} \left(\frac{b-1}{a-1} \right)$$

Then:

If p >1 the DMU is operating under increasing returns to scale
If p = 1 the DMU is operating under Constant returns to scale
If p<1 the DMU is operating under decreasing returns to scale [14]

For a more comprehensive explanation of economies of scale, please watch the following video:
https://youtu.be/JdCgu1sOPDo

3.3 Input or Output Orientation

A DEA model can be input or output oriented:

- Under input orientation, DEA minimizes input for a given output; in other words, it indicates how much a DMU should reduce the level of its inputs for the given level of outputs
- Under output orientation, DEA maximizes output for a given input; in other words, it indicates how much a DMU should increase its level of outputs for the given level of inputs.

Which orientation should be chosen depends on the variables that the decision-maker has control of. For example, in transportation systems, a central authority (ministry, mayor etc.) has control over how many vehicles of public transportation will be available in a region, how many drivers, how much the individual ticket will cost etc. In other words, the decision-maker has control over the inputs and not over the outputs (how many passengers will use the specific mean of public transportation). In such a case, the DEA problem should be solved under input orientation.

In the model of [9] it was assumed that we operate under Constant Returns to Scale. An example of a CRS frontier is given in the figure (figure 3) below.

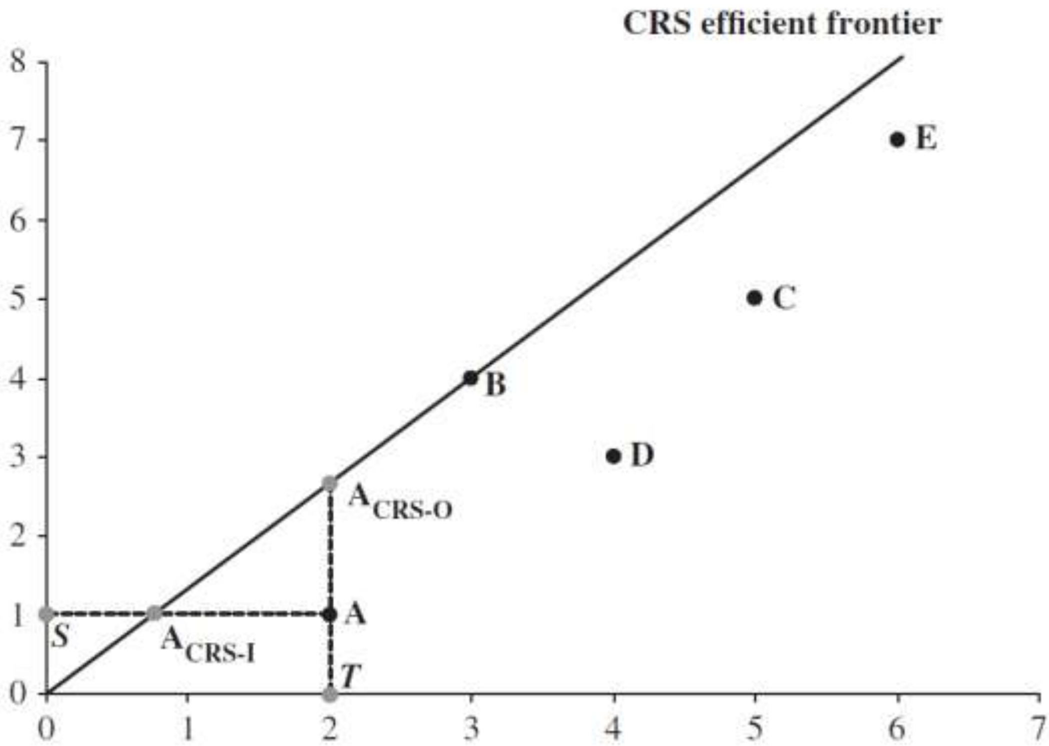


Figure 3. CRS frontier for the evaluation of DMUS [11]

In the model by [10] it was assumed that we operate under Variable Returns to Scale (VRS). An example of a VRS frontier is illustrated in figure 4 below.

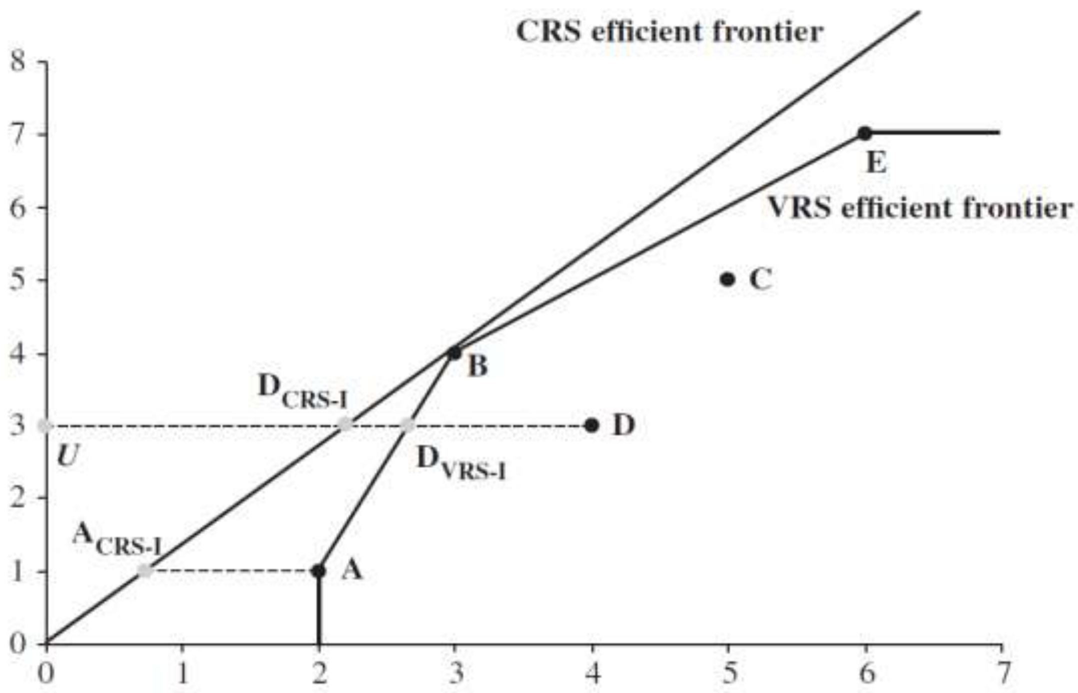


Figure 4. VRS frontier for the evaluation of DMUs[11]

Worker placement board games are a category of games, where each player is required to select individual actions from a set of actions available to all players. These actions are represented by figures, tokens, etc. There is usually a limit in the number of tokens (workers) that a player can place in a single turn in specific places (https://boardgamegeek.com/boardgamemechanic/2082/worker-placement).

Famous worker placement games are:

- *Agricola (https://boardgamegeek.com/boardgame/31260/agricola)*
- *The pillars of the Earth (https://boardgamegeek.com/boardgame/24480/pillars-earth)*
- *Lords of Waterdeep (https://boardgamegeek.com/boardgame/110327/lords-waterdeep)*

For a detailed introduction and history of worker placement games please check the link below:
https://www.youtube.com/watch?v=4E6al0Lx80M

3.3.1 Example

Let us assume that in a potential board game there are 5 mines in which each player must place some workers. Looking at the figure 3 above (which corresponds to one player), the mines are represented by the nodes labeled A to E. The x-axis represents the number of workers (input) and the y-axis the kilograms (kg) of precious metal that they excavated in each mine (output). For example, in mine A, 2 workers have been placed and the excavated 1 kg of precious metals, in mine B 3 workers have been placed that excavated 4 kg of precious metals etc.
To evaluate the efficiency of each mine all that is necessary is to check its position with regards to the frontiers and contemplate on the nature of scales. To identify the nature of scale, focus on the slope

of the VRS efficient frontiers (or the projection of VRS inefficient points). In figure 5 below, the projections of mines D and C in the VRS frontier are the points $D_{vrs\text{-}I}$ and $C_{vrs\text{-}i.}$

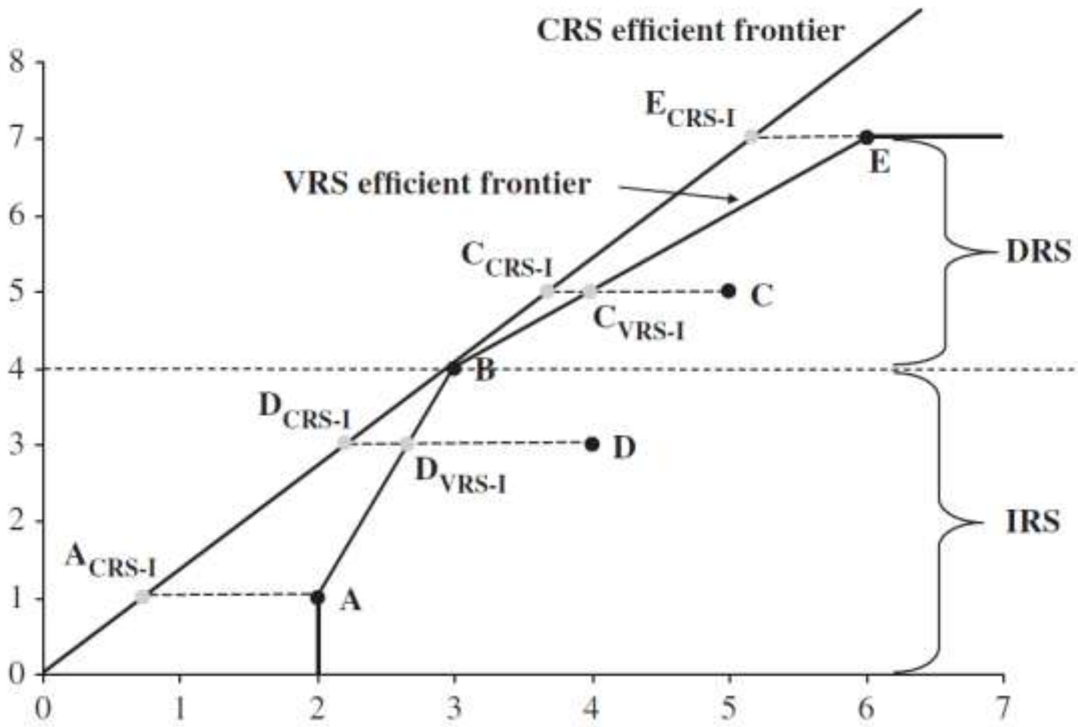


Figure 5. Slopes, nature of returns to scale and projections [11]

To calculate the slopes, the following equations are necessary:

$$\frac{change\ in\ y}{change\ in\ x} = \frac{y_i - y_o}{x_i - x_o}$$

For example, the slope of $D_{vrs\text{-}I}$ is:

$$\frac{y_i - y_o}{x_i - x_o} = \frac{3-0}{2.67-0} = 1.1$$

While the slope of A on the VRS frontier is:

$$\frac{y_i - y_o}{x_i - x_o} = \frac{1-0}{2-0} = 0.5$$

The two slopes indicate that the productivity of A is inferior to $D_{vrs\text{-}I.}$ The ratio of productivity is increasing with scale (size). Therefore, mines A and D are facing decreasing returns to scale.
On the other hand, the slopes of $C_{vrs\text{-}I}$ and E are:

$$slopeC = \frac{5}{4} = 1.25$$

$$slopeE = \frac{7}{6} = 1.17$$

The slopes indicate that the ratio of productivity is decreasing with scale (size). Therefore, mines C and E are facing decreasing returns to scale.

A few more comments on the example:

- Mine A is located on the VRS frontier, but not on the CRS frontier. This indicates that the efficiency of mine A is because of **inappropriate scale**. A is facing increasing returns to scale, thus it must increase its output.
- Mine B is located both on the VRS and CRS frontiers, indicating that it is efficient.
- Mine C is located neither on the VRS nor on the CRS frontier, indicating that its inefficiency occurs due to **inappropriate scale and poor management**. Mine C is facing decreasing returns to scale.
- Mine D is located neither on the VRS nor on the CRS frontier, indicating that its inefficiency occurs due to **inappropriate scale and poor management**. Mine D is facing decreasing returns to scale.
- Mine E is located on the VRS frontier but not on the CRS frontier, indicating that the inefficiency of mine E occurs because of **inappropriate scale**. Mine E is facing decreasing returns to scale, thus it must decrease its output.

3.3.2 Exercises

What is the main purpose of DEA?

- It measures the technical efficiency of Decision Making Units.
- It measures the productivity of Decision Making Units.
- It measures the profitability of Decision Making Units.

How is efficiency measured?

- By the ratio of input over output
- By the ratio of output over input
- By the ratio of weighted input over weighted output
- By the ratio of weighted output over weighted input

In Constant Returns to Scale:

- Output increases by the same proportional increase in input
- Output decreases while input increases
- Output increases while input decreases

A decision-maker must evaluate the bus lines that circulate in a city. His only decision will be to decide (based on his assessment) how many drivers each bus line must employ. To assess the performance of the bus lines, the decision-maker must use:

- An input-oriented DEA model
- An output-oriented DEA model

The same decision-maker has gathered the data and has seen that the 5 bus-lines of the city (named after their destination station) have the following characteristics:

- Hogwarts Bus line: Employs 20 drivers and services 100 people
- Paddington Station: Employs 30 drives and services 400 people
- King's Landing Bus line: Employs 50 drivers and services 500 people
- Mordor Line: Employs 40 drivers and services 300 people
- Gondor Station: Employs 60 drivers and services 700 people

Which bus line is the most efficient?

- Hogwarts Bus Line?
- Paddington Station?
- King's Landing Bus Line?
- Mordor Line?
- Gondor Station?

What should happen in order to increase the efficiency of the Gondor Station?

- Increase the number of people that are serviced (more advertising etc.)
- Decrease the number of people that are serviced
- Leave it as it is

3.4 Mathematical Modeling

So far, the basics notions of DEA have been described along with an approach to calculating efficiency by graphical means. However, this qualitative approach is limiting; if the number of DMUs increases or in a case of multi-input and multi-output the graphs cannot help to identify efficiency. For that reason, there is the need to define the DEA method mathematically. Data Envelopment Analysis is approached as a Linear Programming problem [14]; [15]; [16]; [17]; [18]

Let's assume that there are N DMUs using n inputs to produce s outputs under Constant Returns to Scale (CRS). We denote x_{ij}(i=1…m, j=1…N) the level/value of the i^{th} input of DMU j, and y_{rj}(r= 1…s, j = 1…N) the level/value of DMU. Then the calculation for the *technical input efficiency* can be found by solving the LP problem [P.1]:

minimize $$\Theta_0 - e\left(\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+\right)$$

subject to the *constraints*:

$$\sum_{j=1}^{N} \lambda_j * x_{ij} = \Theta_0 * x_{ij_o} - S_i^-, i = 1 \dots m$$

$$\sum_{j=1}^{N} \lambda_j * y_{rj} = y_{rj_o} + S_r^+, r = 1 \dots s$$

$$\lambda_j \geq 0, j = 1 \dots N, S_r^+, S_i^- \geq 0$$

The technical input efficiency of $DMUj_0$ is Θ_0. The character e is a very small value. In practice the model is solved in two stages. In the first stage, Θ_0 is minimized ignoring the slack[3] variables S_i^- and S_r^+.Thus, we get the technical input efficiency for $DMUj_0$, denoted as Θ_0^*. We substitute the value to the above model and re-solve it but this time with the objective of *maximizing*

$\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+$.

The second stage calculates an optimal set of slacks, while ensuring that we have the optimal technical input efficiency in the calculations. The results from solving the LP problem can be:

- $DMUj_0$ is Pareto-efficient[4]if and only if Θ_0^* = 1 andS_i^- and S_r^+ = 0 for all inputs and outputs.
- If one of the slack values is positive at the optimal solution of the model, it means that the corresponding input (or output) of $DMUj_0$ can be further improved.
- If none of the above applies, then $DMUj_0$ has technical input efficiency equal to Θ_0^*.

3.4.1 Example of Assessing Metro Lines

A company operates 4 metro lines transporting people. The following table (table 5) summarizes the data for one quarter.

Table 5. Metro line costs

Metro Line	Driver Hours (100 hr)	Operational Costs (10000 €)	Passengers (per 10000)
1	4.1	2.3	90
2	3.8	2.4	100
3	4.4	2.0	95
4	3.4	3.4	120

The inputs are driver hours and Operational costs while the output is the passengers serviced. Hence, if we wish to form the DEA model to measure technical input efficiency for Metro Line 1:

[3] In Linear Optimization, a slack variable is one that is added to an inequality constraint to transform it into an equality. In the solution of the problem, if a slack variable is zero then the constraint is binding, if a slack variable is positive the constraint is non-binding and if a slack variable is negative then the constraint is violated, and the solution is not feasible

[4] If a solution is Pareto efficient, no changes can be made in one criterion without worsening any other

minimize $$\Theta_1 - e(S^-_{drivers} + S^-_{Cost} + S^+_{Passengers})$$

subject to the *constraints*:

$$4.1 * \Theta_1 - S^-_{drivers} = 4.1 * \lambda_1 + 3.8 * \lambda_2 + 4.4 * \lambda_3 + 3.4 * \lambda_4$$
$$2.3 * \Theta_1 - S^-_{Costs} = 2.3 * \lambda_1 + 2.4 * \lambda_2 + 2.0 * \lambda_3 + 3.4 * \lambda_4$$
$$90 + S^+_{Passengers} = 90 * \lambda_1 + 100 * \lambda_2 + 95 * \lambda_3 + 120 * \lambda_4$$
$$\lambda_j \geq 0, j = 1,2,3,4$$

The mathematical model and the example are focused on assessing the technical input efficiency of a DMU under Constant Returns to Scale. For the technical output efficiency of a DMU under Constant Returns to Scale the following LP model needs to be solved [P.2]:

maximize $$h_0 + e\left(\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+\right)$$

subject to the *constraints*:
$$\sum_{j=1}^{N} \lambda_j * x_{ij} = x_{ij_o} - S_i^-, i = 1 \dots m$$
$$\sum_{j=1}^{N} \lambda_j * y_{rj} = h_0 * y_{rj_o} + S_r^+, r = 1 \dots s$$
$$\lambda_j \geq 0, j = 1 \dots N, S_r^+, S_i^- \geq 0$$

The technical output efficiency of DMU_{jo} is:
$$\frac{1}{h_0}$$

The slack variables of S_i^- and S_r^+represent any additional increase and/or input reductions that are feasible. The results from solving the LP problem can be:

- $DMU_{jo}$is Pareto-efficient if h_o^*= 1 and S_i^- and S_r^+= 0
- DMU_{jo}has technical output efficiency of $1/h_o^*$

Finally, it should be noted that technical input efficiency and technical output efficiency of a DMU under Constant Returns to Scale are equal:
$$\Theta_o{}^* = \frac{1}{h_0{}^*}$$

3.4.2 Example of Assessing Metro Lines (continued)

From the previous example, if we wish to formulate the LP model to assessing the technical output efficiency of Line 1:

$$\text{maximize} \qquad h_1 + e\left(S^-_{drivers} + S^-_{Cost} + S^+_{Passengers}\right)$$

subject to the *constraints*:

$$4.1 - S^-_{drivers} = 4.1 * \lambda_1 + 3.8 * \lambda_2 + 4.4 * \lambda_3 + 3.4 * \lambda_4$$
$$2.3 - S^-_{Costs} = 2.3 * \lambda_1 + 2.4 * \lambda_2 + 2.0 * \lambda_3 + 3.4 * \lambda_4$$
$$90 * h_1 + S^+_{Passengers} = 90 * \lambda_1 + 100 * \lambda_2 + 95 * \lambda_3 + 120 * \lambda_4$$
$$\lambda_j \geq 0, j = 1,2,3,4$$

3.4.3 Exercises

1) Formulate the models (for the previous example) for:

- Technical Input efficiency of Metro Line 2
- Technical Output efficiency of Metro Line 2
- Technical Input efficiency of Metro Line 3
- Technical Output efficiency of Metro Line 4

2) A taxi owner wishes to evaluate the performance of five of the drivers that he employs in his taxi and has gathered the following data.

Table 6. Problem data

Driver	Hours worked (100 hr)	Passengers Serviced during these hours	Revenues gathered (100 €)
A	4.2	500	17.5
B	4.0	350	17.0
C	4.3	800	24.2
D	4.7	530	18.9

i. What can be considered the inputs and outputs of the evaluation
ii. By employing the DEA method, you must choose if you wish to evaluate the input or output efficiency. Which one would you use in the particular case?
iii. Formulate the LP model to calculate the performance of:
   a. Driver A
   b. Driver D

3.5 DEA under Variable Returns to Scale

The evaluations that were performed thus far, assumed that the DMUs operated under Constant Returns to Scale. However, as it was seen before, this may be an optimistic assumption. As a result, we need to calculate the efficiency under Variable Returns to Scale (VRS).

The VRS DEA model does not differ much from their CRS counterparts. For assessing the technical input efficiency of a DMU under Variable Returns to Scale (VRS), the following LP problem needs to be solved [P.3]:

$$minimize \ \ \Theta_0 - e\left(\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+\right)$$
$$subject\ to\ the\ constraints:$$

$$\sum_{j=1}^{N} \lambda_j * x_{ij} = \Theta_0 * x_{ij_o} - S_i^-, i = 1 \dots m$$
$$\sum_{j=1}^{N} \lambda_j * y_{rj} = y_{rj_o} + S_r^+, r = 1 \dots s$$
$$\sum_{j=1}^{N} \lambda_j = 1$$
$$\lambda_j \geq 0, j = 1 \dots N, S_r^+, S_i^- \geq 0$$

The only difference is that another constraint was added, the one which the sum of the λ parameters equals to 1. This means that the efficiency frontier does not take the form of Figure 3, but that of the piecewise linear form of Figures 4 and 5. The results of the P.3 model can yield:

- DMU is Pareto-efficient if Θ_0^* is equal to 1 and the slack variables S_i^- and S_r^+ are equal to zero. The technical efficiency of a DMU under Variable Returns to Scale is called *pure technical efficiency,* to separate it from its CRS counterpart.
- DMU has pure technical efficiency of Θ_0^*.
- The pure technical efficiency of a DMU cannot be less than its technical input efficiency [14].

3.5.1 Example of Assessing Metro Lines

A company operates 4 metro lines transporting people. The following table (table 7) summarizes the data for one quarter.

Table 7. Metro line data

Metro Line	Driver Hours (100 hr)	Operational Costs (10000 €)	Passengers (per 10000)
1	4.1	2.3	90
2	3.8	2.4	100
3	4.4	2.0	95
4	3.4	3.4	120

i. Define the inputs and the outputs of the DEA model
ii. Write the LP model to calculate the pure technical input efficiency of Metro Line 3

Similarly, to the CRS DEA models, apart from the pure technical input efficiency, the technical output efficiency can be calculated by solving the following LP model [P.4]:

$$maximize\ h_0 + e\left(\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+\right)$$

subject tothe constraints:

$$\sum_{j=1}^{N} \lambda_j * x_{ij} = x_{ij_o} - S_i^-, i = 1 \dots m$$
$$\sum_{j=1}^{N} \lambda_j * y_{rj} = h_0 * y_{rj_o} + S_r^+, r = 1 \dots s$$
$$\sum_{j=1}^{N} \lambda_j = 1$$
$$\lambda_j \geq 0, j = 1 \dots N, S_r^+, S_i^- \geq 0$$

The technical output efficiency of $DMU_{jo}$is

$$\frac{1}{h_0}$$

The P.4 model can yield the following results:

- DMU is Pareto-efficient if h_0^*=1 and the slack variables S_i^- and S_r^+ are equal to zero.
- DMU haspure technical output efficiency of

$$\frac{1}{h_o^*}$$

The pure technical output efficiency cannot be less than the technical output efficiency of model P.2[14].

3.5.2 Exercise

A taxi owner wishes to evaluate the performance of five of the drivers that he employs in his taxi and has gathered the following data (table 8).

Table 8. Taxi problem data

Driver	Hours worked (100 hr)	Passengers Serviced during his hours	Revenues gathered (100 €)
A	4.2	500	17.5
B	4.0	350	17.0
C	4.3	800	24.2
D	4.7	530	18.9

Formulate the LP model to calculate the pure technical input and output efficiencies of:

a. Driver A
b. Driver D

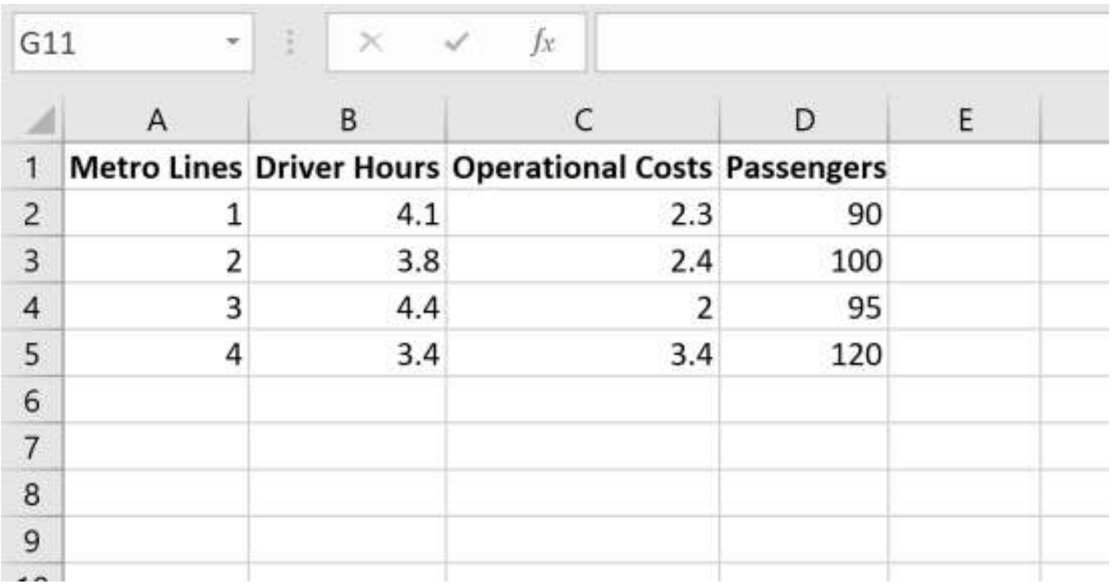
3.6 How to solve a DEA model with Excel

Thus far the theory, notations and how to formulate a DEA model under Constant and Variables Returns to Scale have been explained. In this section, there will be a presentation on how to solve the particular models using Excel. As an example, the example of the Metro Lines will be used. The data for the 4 metro lines are as in table 9.

Table 9. Problem data

Metro Line	Driver Hours (100 hr)	Operational Costs (10000 €)	Passengers (per 10000)
1	4.1	2.3	90
2	3.8	2.4	100
3	4.4	2.0	95
4	3.4	3.4	120

The first thing is to formulate the LP model in Excel;thus we transfer the above table to an Excel Spreadsheet like the picture (Figure 6) below.



	A	B	C	D	E
1	Metro Lines	Driver Hours	Operational Costs	Passengers	
2	1	4.1	2.3	90	
3	2	3.8	2.4	100	
4	3	4.4	2	95	
5	4	3.4	3.4	120	
6					
7					
8					
9					

Figure 6. Data to formulate the DEA model

Following, we define a column for the λ values (blue circle) for which we need to set the initial values. Furthermore, we define the objective function (red circle) and the DMU under evaluation (green circle, figure 7).

A13 | Constraints

	A	B	C	D	E	F	G	H
1	Metro Lines	Driver Hours	Operational Costs	Passengers	λ			
2	1	4.1	2.3	9	1			
3	2	3.8	2.4	10	0			
4	3	4.4	2	9.5	0			
5	4	3	3.4	12	0			
6								
7						DMU under evaluation	1	Efficiency
8								0

Figure 7. Definition of objective function and DMU under evaluation

In the next step we formulate the constraints according to figure 8 below.

	A	B	C	D	E	F	G	H
1	Metro Lines	Driver Hours	Operational Costs	Passengers	λ			
2	1	4.1	2.3	9	1			
3	2	3.8	2.4	10	0			
4	3	4.4	2	9.5	0			
5	4	3	3.4	12	0			
6								
7						DMU under evaluation	1	Efficiency
8	Constraints							0
9	Driver Hours	4.1	<=	0				
10	Operational Costs	2.3	<=	0				
11	Passengers	9	>=	9				
12	Σλ	1	=	1				
13								
14								
15								

Figure 8. Definition of constraints

In the respective cells we set:

- Cell B9: =SUMPRODUCT(B2:B5,$E$2:$E$5)
- Cell B10: =SUMPRODUCT(C2:C5,$E$2:$E$5)
- Cell B11: =SUMPRODUCT(D2:D5,$E$2:$E$5)
- Cell B12: =SUM(E2:E5)

Then, we define the right-hand sides of the constraints as following:

- Cell D9: =$H$8*INDEX(B2:B5,G7,1)
- Cell D10:=$H$8*INDEX(C2:C5,G7,1)
- Cell D11:=INDEX(D2:D5,G7,1)
- Cell D12: =1

Finally, to solve the model we need to use Excel's solver add-in. To Do so:

i. Click the **File** tab and then click **Options**
ii. Click **Add-ins** and the in the **Manage** box select the **Excel Add-ins** option
iii. Click **Go**
iv. In the Add-ins available,select the Solver Add-in and then OK.
v. Go to **Data** tab and select the Solver option.

The following window will pop-up (figure 9).

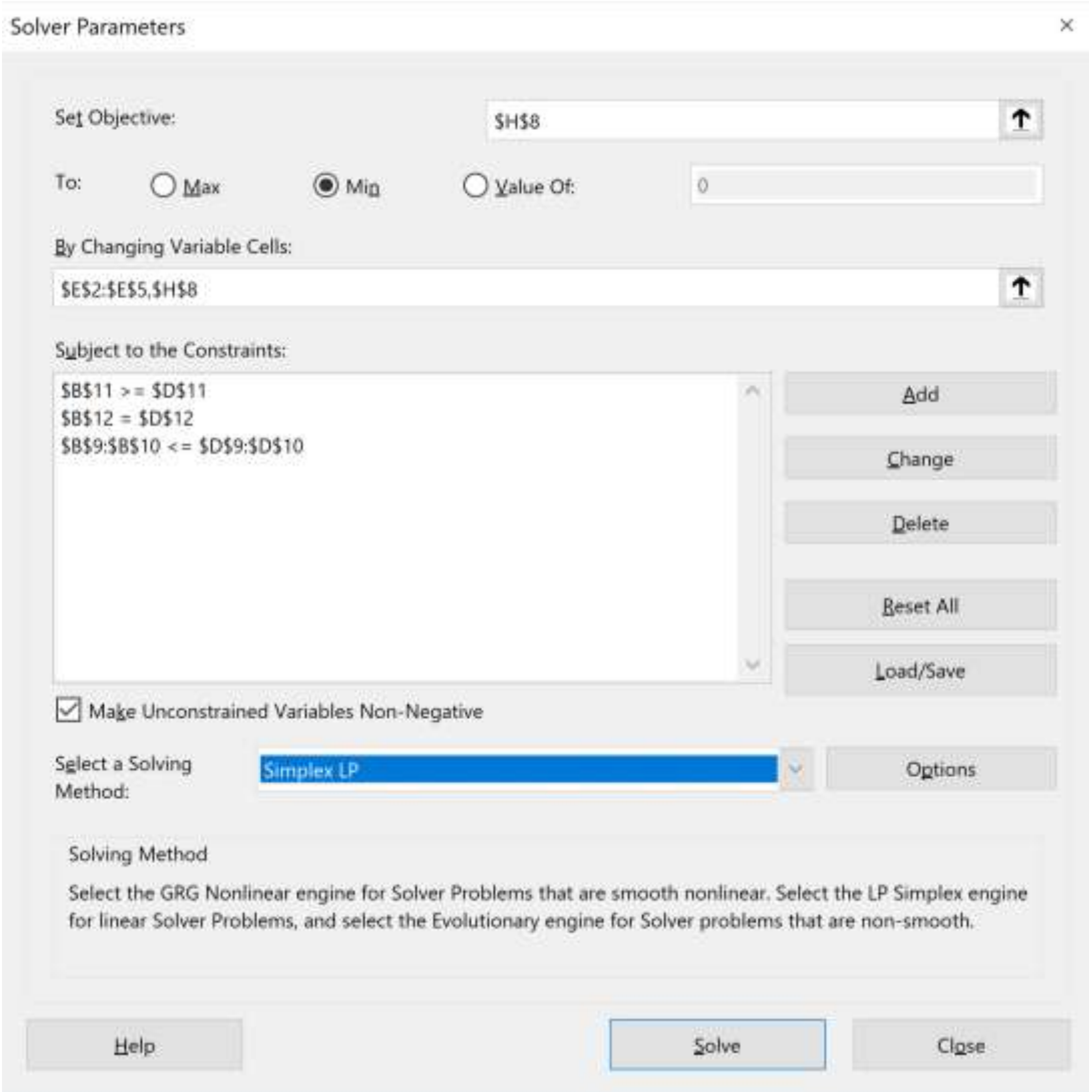


Figure 9. Solver parameters setting

After you have copied the values above, press Solve. The solution to the model is displayed in cell H8.

3.6.1 Exercises

1)Calculate the efficiencies of the remaining metro lines. Which one is the most and which one is the least efficient?

2) The model that was described above was:

i. A DEA CRS input model?
ii. A DEA CRS output model?
iii. A DEA VRS input model?
iv. A DEA VRS output model?

3) A taxi owner wishes to evaluate the performance of five of the drivers that he employs in his taxi and has gathered the following data (table 10).

Table 10. Exercise data

Driver	Hours worked (100 hr)	Passengers Serviced during his hours	Revenues gathered (100 €)
A	4.2	500	17.5
B	4.0	350	17.0
C	4.3	800	24.2
D	4.7	530	18.9

i. Solve the problem in Excel for all Drivers. Assume Constant Returns to Scale
ii. Talk about the Interpretation of the results of DEA models

4. Multiple Criteria Decision Aid

Multiple Criteria Decision Aid (MCDA) is an evolving area of operations research that aims to provide the decision-maker with the tools necessary in solving complex decision problems, where often contradictory and competitive aspects should be considered. The main objective of MCDA and a common element of all methodological approaches in this domain is the development and use of synthesis patterns of all the basic parameters of a problem so as to provide support in making rational decisions on the basis of its system of values and preferences.

Multiple criteria aid approachesdiffer in the way they combine the elements of the problem in hand. Formal multiple criteria aid techniques usually provide a specific system of relative gravity for different criteria. The key role of these techniques is to address the difficulties that decision-makers seem to have to handle consistently and logically large numbers of complex information. They can be used to determine the preferred option, to rank options, to list a limited number of options, to follow a detailed evaluation, or simply to separate acceptable and unacceptable possibilities.

Most decisions that decision-makers need to take are characterized by conflicting criteria and more than one alternatives, thus it is necessary to provide a clear structure of the problem before arriving to its solution. MCDA methods use a series of steps that clarify the elements and boundaries of the problem [19]and consist of a mathematical model that ranks the alternatives.MCDA methods are assumed to provide clear, rational and easily justifiable explanations, can deal with quantitative and qualitative data and can involve the preferences of more than one decision-maker [20].There is a large variety of MCDA methods and in the following sections two of the most popular methods will be described in detail.

4.1 The PROMETHEE method

4.1.1 Introduction

PROMETHEE is an acronym that stands for **P**reference **R**anking **O**rganization **METH**od for **E**nrichment of **E**valuations. PROMETHEE expresses the method of organizing the preference ranking for an enhanced assessment of a problem. Thus, the PROMETHEE method provides a decision, either with a single choice or with alternatives, based on preference grades among the available options [11].The method was developed by Brans, Mareschal, and Vicke [21]; [22]; [23] and it belongs to the outranking family of MCDA methods. The method is considered popular and is used in a variety of research areas. A report from June of 2018 states the total numberof references mentioning any of the variations of PROMETHEE to 1827 papers. In the following two tables (tables 11 and 12), the scientific areas where PROMETHEE is more popular, and also how its popularity has grown the last years is presented [24].

Table 11. Delivery of papers on PROMETHEE by application areas [21]

Area	N	%
Environment management	366	20
Services and/or public applications	343	18.80
Industrial applications	278	15.20
Energy management	165	9
Water management	121	6.60
Finance	107	5.90
Transportaion	81	4.40
Procurement	61	3.30
Other topics	78	4.20

Table 12. Delivery of papers on PROMETHEE by countries [24]

Date	23/11/2015	12/03/2016	03/12/2016	19/03/2017	26/06/2018
Papers	1236	1317	1469	1526	1827
#countries	63	65	68	70	75
%Europe	48.10%	48.10%	46.60%	46.40%	44.90%
Authors	2138	2272	2564	2681	3256

4.1.2 Methodology

The method is characterized by the following steps:

1. Construct the decision matrix with all the alternatives, criteria, preference parameters and weights.
2. Calculate the differences between the evaluations of the actions for each criterion.
3. Construct the pairwise comparison matrix for each criterion.
4. Calculate the unicriterion net flows.
5. Calculate the weighted unicriterion flows.
6. Calculate the global preference net flows.
7. Rank the actions according to PROMETHEE I or II [25].

It is quite important to state that the decision-maker can use both qualitative and quantitative criteria. The final ranking of the alternatives is the result of the global flows. The PROMETHEE method is known as one of the simplest multicriteria methods. There are some visual tools, which make every problem more understandable to the decision-makers. In addition, there are sensitivity analysis tools to analyze the robustness of the final ranking of the alternatives. Furthermore, some software tools are available for free, contributing to the extended use of the method [11]. In the following sections, several important aspects of the method will be clarified, while a numerical, step-by-step example will be presented at the end of the section.

4.1.2.1 Unicriterion preference degrees

The PROMETHEE method is based on the calculation of preference degrees. Preference degrees range between 0 and 1, expressing whether one option is more preferred than another according to a decision-maker's attitudes. Preference degree 1 means a strong or total preference for an action with respect to the specific criterion. If there is no preference, then 0 is the value chosen. For an intermediate state, i.e. when there is a preference but not strong enough, then a value between 0 and 1 is selected.

We use the term pairwise preference degree, since the preference of action A over action B cannot be derived from the preference of action B on action A (and vice versa). The PROMETHEE method will help the decision-maker evaluate these unicriterion pairs of preference degrees. For each criterion, the degree of unicriterion preference is calculated by adjusting or enriching the assessments of the actions through additional preference information.

In multiple criteria decision making an important point is how the decision-maker takes into consideration the differences between ratings of the alternatives (or actions) for every specific criterion. This also plays an important role in PROMETHEE. These comparison pairs are based on the difference between the evaluations of the two actions (i.e. the price difference between the actions). Therefore, the decision-maker can choose between six types of preference functions: Usual, U-shape, V-shape, Level, V-shape with indifference (also known as linear function) and Gaussian function. These preference functions are shown in figure 10 with their definitions and parameters. It is very important for every decision-maker to understand how these functions interact with the data, in order to choose the most appropriate for every problem. For instance, if the decision-maker opts for a linear function (type 5), preferences will gradually increase as a function of the difference between the ratings on a particular criterion. When using the Gaussian preference function (type 6), the increase follows an exponential function [11].

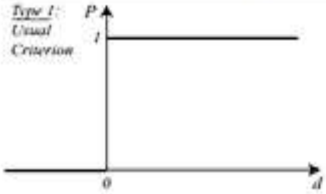
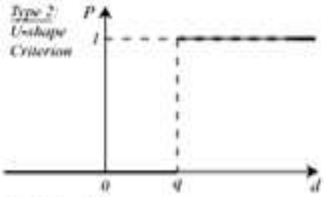
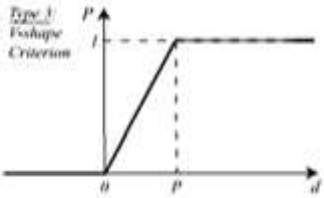
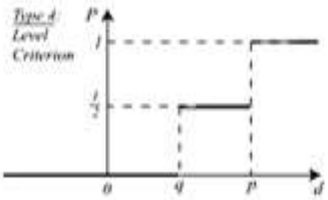
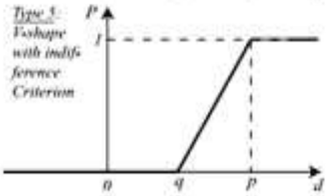
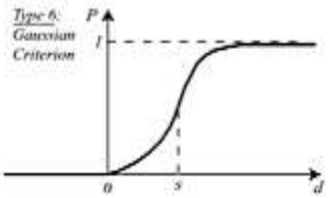
Generalized criterion	Definition	Parameters to fix
	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 & d > 0 \end{cases}$	–
	$P(d) = \begin{cases} 0 & d \leq q \\ 1 & d > q \end{cases}$	q
	$P(d) = \begin{cases} 0 & d \leq 0 \\ \frac{d}{p} & 0 \leq d \leq p \\ 1 & d > p \end{cases}$	p
	$P(d) = \begin{cases} 0 & d \leq q \\ \frac{1}{2} & q < d \leq p \\ 1 & d > p \end{cases}$	p, q
	$P(d) = \begin{cases} 0 & d \leq q \\ \frac{d-q}{p-q} & q < d \leq p \\ 1 & d > p \end{cases}$	p, q
	$P(d) = \begin{cases} 0 & d \leq 0 \\ 1 - e^{-\frac{d^2}{2s^2}} & d > 0 \end{cases}$	s

Figure 10. Types of generalized criteria [26]

For setting each preference mode, one or two parameters are required. The linear preference function requires two parameters: an *indifference threshold q* and the *preference threshold p*. On the other hand, the Gaussian function requires only one parameter: the *inflexion point s*. If the difference between evaluations of a criterion is less than the threshold of indifference q, then no discrepancy between these two actions can be perceived by the decision-maker (i.e. the degree of preference is 0). If the difference is higher than the preference threshold p, then the preference is strong (i.e., the degree of preference is 1). The preference function gives the value of the preference for the differences between the indifference threshold q and the preference threshold p.

It is worth mentioning, that the PROMETHEE method can use not only numerical values, but can easily handle scales (e.g. good, excellent, etc.) and verbal values that can be translated to numerical ones.

4.1.2.2. Unicriterion positive, negative and net flows

The decision-maker needs specific information in order to draw conclusions for the problem under consideration and it is not an easy procedure to understand the ranking of the alternatives through a table with preference degrees or from the graphical representation of the table, especially when the number of actions is large. Therefore, the so-called positive flows, negative flows and net flows are calculated. They measure the way in which any action is preferred over all other actions or the way all the other actions are preferred to it.

Positive flows

The unicriterion positive flow has values between 0 and 1. These values indicate how an action is preferred over all other actions on that criterion. The higher the positive flow, the more preferred the action to all the other actions.

Negative flows

On the other hand, negative flows represent how other actions are preferred against the chosen action. The values also vary between 0 and 1. The lower the negative flow, the more preferred the action over all the other actions is.

Net flows

In the final stage both positive and negative flows are taken into consideration. So, the net flows are calculated by subtracting the negative flows from the positive flows. The values for the net flows range between -1 and 1. The net flows need to be maximized and the reason why is that they express the balance between the global strength and the global weakness of every action[26, 22].

Global flows

In the above section, only one criterion at a time was examined. However, PROMETHEE needs to take into consideration all the criteria together. For that, the relative importance of each criterion needs to be provided by the decision-maker. For example, a car's fuel consumption may be twice more important than its price. On the other hand, price might be more important than the power. In other words, there is a need for the decision-maker to determine a weight for each criterion that allows the aggregation (through a weighted sum) of all positive, negative and net flows of unicriterion to global positive flows, global negative flows and global net flows, taking into account all the criteria at once.

The overall positive score indicates how an action is preferred globally to all other actions when considering all the different criteria. Since weights have been normalized, the global positive score is always between 0 and 1. The higher the value, the most preferred the action is. Accordingly, global negative scores indicate how an action is not preferred when compared with all the other actions. The negative score is always between 0 and 1 the lower the value the more the alternative is preferred over the others. The net global flows are calculated by subtracting the global negative flows from the global positive flows.

4.1.3 The PROMETHEE I partial ranking

There are different variations of the PROMETHEE method. PROMETHEE I offers a partial ranking of the alternatives based on the positive and negative flows. When the flows of two actions need analysis, four different scenarios can be formed:

- An action is better ranked by the other if the overall positive and negative flows are at the same time better (i.e. if the overall positive score is higher and the global negative flow is lower).
- An action has a worse score than another, when positive and negative scores are worse at a global level.
- Two actions are said to be incomparable if an action has a better global positive flow but a worse global negative flow (or vice versa).
- Two actions are called indifferent if they have identical (equal) positive and negative flows.

4.1.4 The PROMETHEE II complete ranking

PROMETHEE II's complete ranking is based only on net flows and leads to a full ranking of all the actions (i.e. the incomparable situation does not exist). The actions are ranked form best to worst.

4.1.5 Gaia plane

The Gaia plane presents a decision problem in a two-dimensional figure, and it attempts to include all aspects of the decision problem: the actions, criteria and information about the decision-maker's preferences (preference parameters and weights). The actions are represented by bullets and the criteria with arrows, and the position of the actions gives the decision-maker a first picture of the existing similarities, for example the closer the actions, the more similar they are. Similarity and non-similarity are determined by the limit of indifference and preference. This means that the Gaia plane depends on the preference information provided by the decision-maker. Accordingly, the relative position of the criteria indicates the correlation and anti-correlation (or conflict) of criteria. The closer the arrows are, the more important are the criteria for the problem. The greater the angle between the criteria, the greater the conflict between them.
The Gaia plane allows the illustration of contradictory views. In addition, the length of the arrow, which represents a criterion, measures its 'discriminating' or 'differentiating' power as a function of

the data. The more different the actions in a criterion, the greater the arrow and, therefore, the most distinctive criterion. The discretionary power of a criterion depends on the selected limitsand the corresponding weight.

Finally, the arrow represented by the letter D, known as the decision stick, illustrates the compromise chosen by the decision-maker as it corresponds to the weight adjustment. The visibility of the actions on this line represents their priorities. The greater the visibility of an action on the stick, the better the action ranks. However, since it is only a two-dimensional representation, it can lead to a loss of information. The amount of information held, so-called delta or D, depends on the data and the number of criteria. As a consequence of the loss of information, the classification resulting from the projection of the decision stick does not necessarily have the exactly the same results with PROMETHEE II [11].

4.1.6 Example- Case study

A big company of the clothing and accessoriesindustry wants to builda new factory in order to strengthen its production activity. After excluding many options, the company has ended up with six final options, taking into account four important criteria:

1. the capital requirements (for the new investment),
2. the available staff (for the new factory),
3. the delivery distance (from the new factory to the customers) and
4. the environmental impact.

The first three criteria are quantitative and fourth is qualitative with values ranging from 1 (worst) to 7 (best). All the criteria need to be maximized. In Table 13 and Table 14, we have all the data for the example. The weights are given hypothetically, and they are not derived with a specific procedure, but they represent the preferences of the hypothetical decision-maker.

Table 13. Data for the numerical example

	Capital Requirements (million €)	**Available Staff (hundred employees)**	**Delivery Distance (hundred kilometers)**	**Environmental Impact (1 to 7)**
Weight	0.4	0.3	0.1	0.2
Option 1	7	8	2	5
Option 2	8	6	5	4
Option 3	9	10	5	10
Option 4	5	4	1	2
Option 5	11	10	3	8
Option 6	9	8	6	7

Table 14. Preference Parameters of all the criteria

Criterion	Function	w_i	q_i	p_i
Capital Requirements (million €)	Usual	0.4	0	0
Available Staff (hundred employees)	Usual	0.3	0	0
Delivery Distance (hundred kilometers)	Usual	0.1	0	0
Environmental Impact (1 to 7)	Usual	0.2	0	0

4.1.6.1 Computation of the Unicriterion net flows and next steps

In the following tables, the reader can study the steps of the methodology in detail. Using the headings from each table,every step of the procedure is depicted.

Table 15. Step 1: Differences between the evaluations of the Options on the criterion 'Capital Requirements'

Difference	Option 1	Option 2	Option 3	Option 4	Option 5	Option 6
Option 1	0	-1	-2	2	-4	-2
Option 2	1	0	-1	3	-3	-1
Option 3	2	1	0	4	-2	0
Option 4	-2	-3	-4	0	-6	-4
Option 5	4	3	2	6	0	2
Option 6	2	1	0	4	-2	0

Table 15 illustrates the differences on the various options on the criterion 'Capital Requirements'. For example, the first row is: Option 1 – Option 1 equals 0 since option 1 cannot be compared with itself. Option 1 – Option 2 = 7 – 8 = -1; Option 1 – Option 3 = 7 – 9 =-2 etc. The other cells of the matrix are calculated similarly.

Table 16 illustrates the differences of the options on the criterion Available staff. For example, the first row is: Option 1 – Option 1 = 8-8 = 0, Option 1 – Option 2 = 8-6 = 2, Option 1 – Option 3 = 8-10 = -2 etc. The other cells in the matrix are calculated similarly.

Tables 17 and 18 show the differences of the evaluations of the various options for the remaining two criteria and they are calculated similar to tables 15 and 16.

Table 16. Step 1: Differences between the evaluations of the Options on the criterion 'Available Staff'

Difference	Option 1	Option 2	Option 3	Option 4	Option 5	Option 6
Option 1	0	2	-2	4	-2	0
Option 2	-2	0	-4	2	-4	-2
Option 3	2	4	0	6	0	2
Option 4	-4	-2	-6	0	-6	-4
Option 5	2	4	0	6	0	2
Option 6	0	2	-2	4	-2	0

Table 17. Step 1: Differences between the evaluations of the Options on the criterion 'Delivery Distance'

Difference	Option 1	Option 2	Option 3	Option 4	Option 5	Option 6
Option 1	0	-3	-3	1	-1	-4
Option 2	3	0	0	4	2	-1
Option 3	3	0	0	4	2	-1
Option 4	-1	-4	-4	0	-2	-5
Option 5	1	-2	-2	2	0	-3
Option 6	4	1	1	5	3	0

Table 18. Step 1: Differences between the evaluations of the Options on the criterion 'Environmental Impact'

Difference	Option 1	Option 2	Option 3	Option 4	Option 5	Option 6
Option 1	0	1	-5	3	-3	-2
Option 2	-1	0	-6	2	-4	-3
Option 3	5	6	0	8	2	3
Option 4	-3	-2	-8	0	-6	-5
Option 5	3	4	-2	6	0	1
Option 6	2	3	-3	5	-1	0

The next step in the method is to calculate the pairwise comparison matrix of the options for all the criteria.

Table 19. Step 2: Pairwise Comparison matrix for the criterion 'Capital Requirements'

	Option 1	Option 2	Option 3	Option 4	Option 5	Option 6
Option 1	0	0	0	1	0	0
Option 2	1	0	0	1	0	0
Option 3	1	1	0	1	0	0
Option 4	0	0	0	0	0	0
Option 5	1	1	1	1	0	1
Option 6	1	1	0	1	0	0

Table 19 shows the pairwise comparison matrix for the criterion 'Capital Requirements'. From table 14 it is known that the preference function for the criterion is the 'Usual'. From figure 9, it can be observed that the 'Usual' function does not require the preference parameters p and q and relies only on the difference in the evaluations. Thus, the preference can be calculated by the function:

$$P(d) = \begin{cases} 0, d \leq 0 \\ 1, d > 0 \end{cases}$$

Where d is the difference in the evaluations. For example, for the first row in table 15 the set of values is {0, -1, -2, 2, -4, 2}. Hence the pairwise preference comparison of Option 1 with all the other options (first row of Table 9) is {0, 0, 0, 1, 0, 0}. Similar calculations are performed for the other options (rest of the rows in table 19).In case that the preference function of the criterion was not the 'Usual', but the 'V-shape' (figure 9), then the preference pairwise comparison matrix would take its values from the function:

$$P(d) = \begin{cases} 0, d \leq q \\ \dfrac{d-q}{p-q}, q < d \leq p \\ 1, d > p \end{cases}$$

For example, for option 1- compared with the other options (first row of table 15), with hypothetical q = 1, p = 3, the values would become: (0, 0, 0.5, 0, 0.5).

Tables 20, 21 and 22 depict the pairwise comparison matrices for the criteria 'Available Staff', 'Delivery Distance' and 'Environmental Impact' respectively.

Table 20. Step 2: Pairwise Comparison matrix for the criterion Available Staff

	Option 1	**Option 2**	**Option 3**	**Option 4**	**Option 5**	**Option 6**
Option 1	0	1	0	1	0	0
Option 2	0	0	0	1	0	0
Option 3	1	1	0	1	0	1
Option 4	0	0	0	0	0	0
Option 5	1	1	0	1	0	1
Option 6	0	1	0	1	0	0

Table 21. Step 2: Pairwise Comparison matrix for the criterion Delivery Distance

	Option 1	**Option 2**	**Option 3**	**Option 4**	**Option 5**	**Option 6**
Option 1	0	0	0	1	0	0
Option 2	1	0	0	1	1	0
Option 3	1	0	0	1	1	0
Option 4	0	0	0	0	0	0
Option 5	1	0	0	1	0	0
Option 6	1	1	1	1	1	0

Table 22. Step 2: Pairwise Comparison matrix for the criterion Environmental Impact

	Option 1	**Option 2**	**Option 3**	**Option 4**	**Option 5**	**Option 6**
Option 1	0	1	0	1	0	0

Option 2	0	0	0	1	0	0
Option 3	1	1	0	1	1	1
Option 4	0	0	0	0	0	0
Option 5	1	1	0	1	0	1
Option 6	1	1	0	1	0	0

The next step consists of calculating the unicriterion net flows. For option 1 for example, the positive net outranking flow is calculated by summing row 1 of table 19 and divide the result by the number of options minus 1, since option 1 cannot be compared with itself. As a result, the positive outranking flow for option 1 for the criterion 'Capital Requirements' is:

$$\frac{0+0+0+1+0+0}{6-1} = 0.2.$$

The negative outranking flow for option 1 (for the criterion 'Capital requirements') is calculated by summing the elements of the first column of table 19and divide the result by the number of options minus 1. As a result, the negative outranking flow for option 1 for the criterion 'Capital Requirements' is:

$$\frac{0+1+1+0+1+1}{6-1} = 0.8.$$

The net flow is calculated by simply subtracting the negative outranking from the positive one: 0.2-0.8 = -0.6. The calculations for the other options are performed in a similar manner. Table 23 below illustrates the net flows for all the options for all the criteria.

Table 23. Computation of the Uniriterion Net Flows

Options	Capital Requirements Net Flows	Available Staff Net Flows	Delivery Distance Net Flows	Environment Impact Net Flows
Option 1	-0.6	0	-0.6	-0.2
Option 2	-0.2	-0.6	0.4	-0.6
Option 3	0.4	0.8	0.4	1
Option 4	-1	-1	-1	-1
Option 5	1	0.8	-0.2	0.6
Option 6	0.4	0	1	0.2

The next step consists of multiplying the unicriterion net flows with the respective weights for each criterion (table 15). Finally, the total net flows are calculated by summing the elements of each row of table 24.

Table 24. Computation of the weighted unicriterion flows

Options	Capital Requirements Net Flows	Available Staff Net Flows	Delivery Distance Net Flows	Environment Impact Net Flows
Option 1	-0.24	0	-0.06	-0.04
Option 2	-0.08	-0.18	0.04	-0.12
Option 3	0.16	0.24	0.04	0.2
Option 4	-0.4	-0.3	-0.1	-0.2
Option 5	0.4	0.24	-0.02	0.12
Option 6	0.16	0	0.1	0.04

Table 25. Computation of the total net flows

Options	Total Net Flows
Option 1	-0.34
Option 2	-0.34
Option 3	0.64
Option 4	-1
Option 5	0.74
Option 6	0.3

Table 25 provides a total ranking, thus PROMETHEE II is used. For a PROMETHEE I ranking we would have stopped at table 13 because the positive and negative outranking flows are necessary.Table 25 informs that Option 5 ranks the best, while Option 4 appears to be the worst of the set.

4.1.7 Visual PROMETHEE

Visual PROMETHEE[5] can assist in graphically depicting the positive and negative outranking flows for all the options (PROMETHEE I). Figure 11 illustrates the flows for the six options of the case study.

[5] http://www.promethee-gaia.net/software.html

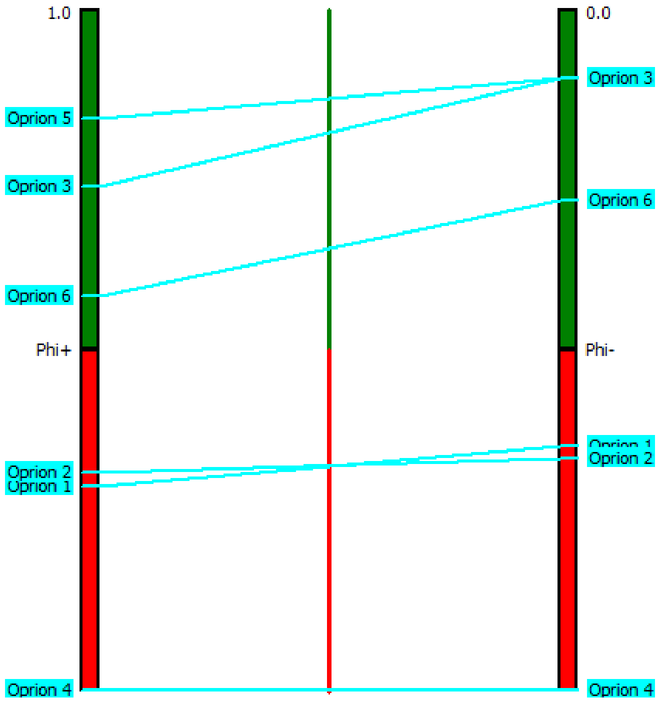


Figure 11. PROMETHEE I Partial Ranking

As it can be observed, Option 5 has the best positive outranking flow (Phi+ bar) – max value and the best negative outranking flow (Phi- bar)- min value. Therefore, Option 5 is the best option in the ranking and Option 3 closely follows, with Option 4 having the worst performance.

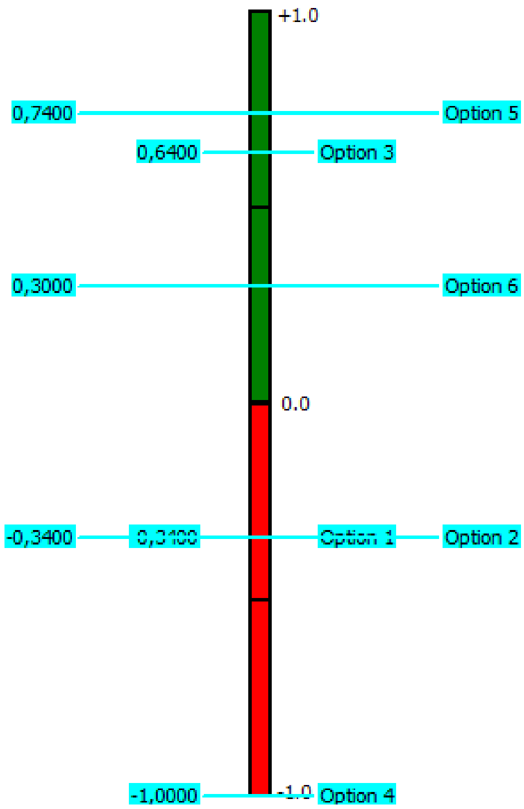


Figure 12. PROMETHEE II Complete Ranking

Figure 12 illustrates the PROMETHEE II complete ranking from the Visual PROMETHEE software and figure 13 summarizes the results.

Rank	action	Phi	Phi+	Phi-
1	Option 5	0,7400	0,8400	0,1000
2	Option 3	0,6400	0,7400	0,1000
3	Option 6	0,3000	0,5800	0,2800
4	Option 1	-0,3400	0,3000	0,6400
5	Option 2	-0,3400	0,3200	0,6600
6	Option 4	-1,0000	0,0000	1,0000

Figure 13. PROMETHEE Flow Table (figure from Visual PROMETHEE)

4.2.8 GAIA Analysis

As it was mentioned in a previous section, the GAIA plane can be also used to provide further insights on the behavior of the various options under all the criteria. Figure 14 illustrates an instance of the GAIA plane for the case study.We can see the options and our selected criteria, which are represented by bullets and arrows, respectively. The available options are not quite similar, due to the fact that they are not close the one to another. On the other hand, the criteria 'Capital Requirements', 'Environmental Impact' and 'Available Staff' are the most important criteria for our problem, as the closer the arrows are, the more important are the criteria for the problem. We also see that the decision stick, which illustrates the compromise solution, is more close to the Option 3, while the Option 3 is second on the ranking of the figure 12, and the Option 5 is a little more distant from the decision stick, while it is first on the ranking of the figure 12. This happens due to the loss of information of the Gaia plane, as it is already mentioned on the corresponding theoretical section.

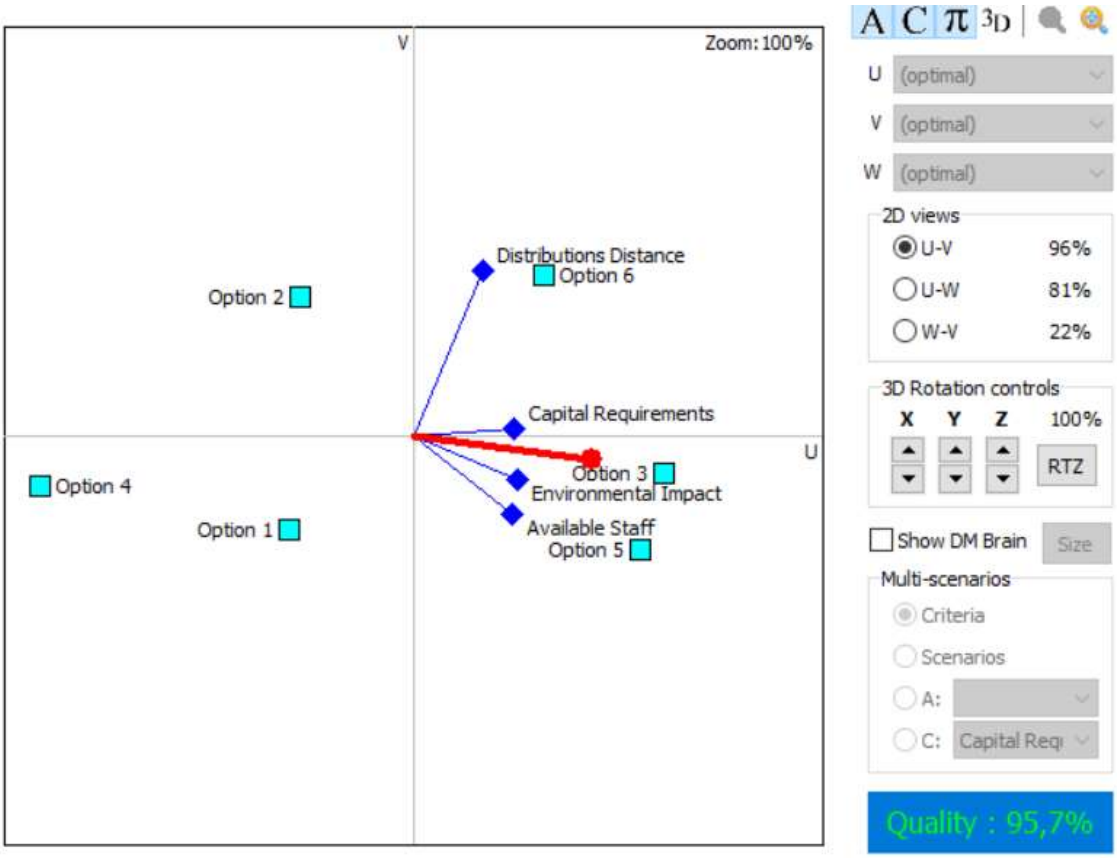


Figure 14. GAIA Visual Analysis (figure from Visual PROMETHEE)

4.2.9 PROMETHEE exercises

A transport company wants to locate its new terminal bus station in a city. There are some criteria that should be taken into consideration, in order to find the best location among the available options. The available options are the locations in the city center, west of the city and east of the city. The criteria that the transport company has set are the following:

1. infrastructure costs (hundreds €)
2. distance from the city center (kilometers)
3. distance from petrol stations (kilometers)
4. traffic level (1 to 7)
5. social impact (1 to 7)

All the criteria need to be maximized.

Table26. Data for the exercise

	Infrastructure Costs (hundreds€)	Distance of the City Center (kilometers)	Distance of petrol stations (kilometers)	Traffic Level (1 to 7)	Social Impact (1 to 7)
Weight	0.4	0.2	0.2	0.1	0.1
City Center	8	3	6	7	5
West	6	15	2	3	2
East	7	26	5	4	2

Table 26. Preference Parameters of all the criteria (Usual)

Criterion	Function	wi	qi	pi
Infrastructure Costs (hundreds€)	Usual	0.4	0	0
Distance of the City Center (kilometers)	Usual	0.2	0	0
Distance of petrol stations (kilometers)	Usual	0.2	0	0
Traffic Level (1 to 7)	Usual	0.1	0	0
Social Impact (1 to 7)	Usual	0.1	0	0

a) Tables 16 and 17 provide the available data and information. Find out what is the best location for the company in order to place its terminal station.

b) The preference functions of the various criteria followed the 'Usual' function, so no preference parameter was necessary. Solve the same problem with different preference functions, as provided in Table 18. Does the ranking change?

Table 27. Preference Parameters of all the criteria (U-shape & Linear)

Criterion	Function	wi	qi	pi
Infrastructure Costs (hundreds€)	U-shape	0.4	2	-
Distance of the City Center (kilometers)	U-shape	0.2	11	-
Distance of petrol stations (kilometers)	U-shape	0.2	4	-
Traffic Level (1 to 7)	Linear	0.1	2	3
Social Impact (1 to 7)	Linear	0.1	3	3

Multiple-choice questions

PROMETHEE stands for

- PReference MEasurement METod Evaluation?
- Preference RankingTHEory with Enriched Evaluation?
- PROfessional THEory for Easy Evaluation?
- Preference Ranking Organization METHod of Enrichement Evaluation?

The main purpose of PROMETHEE is to

- assign criteria to actions?
- prioritize actions based on criteria and constraints?
- rank actions based on criteria.
- assign goals to alternatives?

Pairwise comparisons are based on which type of scale?

- Nominal scale
- Interval scale
- Ratio scale
- Ordinal scale

Here you can find auseful link explaining the Visual PROMETHEE software.
https://www.youtube.com/watch?v=qQJO2UYJRc8

4.3 Analytic hierarchy process

4.3.1 Introduction

The Analytic Hierarchy Process (AHP) is a technique developed by Saaty [27]; [28]. The development of the method was a reflection of the lack of a practical and easily applicable method for setting priorities, and decision-making. It is a technique for dealing with complex decision problems, based on mathematics and human psychology, and is designed to combine logic, feelings and intuition into an effective framework for solving multi-criteria problems. The AHP methodology is based on a group of axioms that clearly define the scope of a problem, represent its structure, quantify its information, relate the individual elements of the problem to later objectives, and evaluate alternatives.

AHP does not require judgments to be consistent or transient, the degree of consistency is communicated to the decision-maker, and it is he who will decide whether the rating of alternatives is credible. The method addresses the problem of weights in a set of activities according to their degree of importance. For this purpose, binary comparisons are made, and a preference scale is developed between the activities based on the estimates of the decision-makers. This process results in the creation of a weight table and an estimate table for each criterion. The initial problem is broken down into individual segments or variables, the variables are hierarchically classified by giving numerical values to the relative significance estimates, and finally, the composition of the estimates is made to determine which variable has the highest priority - influence on the result [29].

Although AHP has been criticized from a methodological point of view, it has been established today as one of the most applied decision-making techniques, and its diffusion is due to its simplicity, clarity and ease of realization. Hierarchical Analysis of Decisions is ultimately a decision-making method for complicated multiple criteria problems that can be used without requiring special knowledge while allowing decision-makers to combine their experience, knowledge, and intuition. The best way to describe the method is by describing its four key functions:

a. the hierarchical analysis of the decision problem in decision elements,
b. the collection of preferences by the decision-maker on decision elements,
c. the calculation of individual priorities for the decision elements;
d. the composition of the individual priorities in the general priorities of the alternatives [11].

There are plenty of AHP scholar works and applications in the literature. The method due to its good results and reliability has received many modifications, for example the Fuzzy AHP [30, 31, 32] and group decision making [33, 34, 35]. In table 29 the scientific areas where AHP is most popular is depicted, where there are selected 150 papers, and are analyzed for their application area, as a glimpse of the most popular publications of AHP [36].

Table 28. Delivery of papers on AHP by application areas [32]

Area	N	%
Engineering	26	17.30
Personal	26	17.30
Social	23	15.30
Manufacturing	18	12
Industry	15	10
Government	13	8.70
Education	11	7.30
Political	6	4
Other topics	12	8

4.3.2 Methodology

4.3.2.1 Problem structuring

AHP is known through a phrase: divide and conquer. Every problem usually is complex, and in MCDA, it is simpler to divide the problem in smaller pieces and solve one sub-problem at each step. AHP has the characteristic of the division during:

1. the structuring of the problem and
2. the elicitation of the priorities through pairwise comparisons [11].

In AHP, there are no numerical judgments, but the decision-maker creates a table with the comparative value of one alternative over another, making the comparisons for all the alternatives. The following equation will help the reader to understand more easily the way that the pairwise comparisons are structured [35].

$$X = \begin{bmatrix} 1 & x_{12} & \dots & \dots & \dots & x_{1n} \\ \frac{1}{x_{12}} & 1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & x_{ij} & \dots & \dots \\ \dots & \dots & \frac{1}{x_{ij}} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 & \dots \\ \frac{1}{x_{1n}} & \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

Where $x_{ij}$is the comparison between element (alternatives or criteria)*i* and *j*:

$$x_{ij} = \frac{1}{x_{ji}} \text{ and } x_{ji} = \frac{1}{x_{ij}}$$

It can be observed that for the diagonal of the matrix $x_{ii} = 1$ as each alternative (or criterion)is equally important to itself. In addition, when the matrix *X* is perfectly consistent, we have the transitivity rule for every comparison [35]:

$$x_{ij} = x_{ik} x_{kj}$$

4.3.2.2 AHP steps

The following paragraphs present a step-by-step explanation of AHP.

Step 1. Pairwise Comparison matrix of the criteria

The first step for AHP is the definition of how two criteria can be compared to each other. The decision-maker expresses his/her preferences. The function

$$\frac{n^2 - n}{2}$$

expresses how many comparisons are needed for the pairwise comparison matrix (where n is the number of criteria) and it leads to a nxn table. A 1 to 9 scale has been proposed by Saaty (1977) and it is used in most of the publications (table 30). A smaller scale may lead the decision-makers to a confusion with regard to the psychological part of the problem, as the differences would not be so clear. However, the decision-maker can use other scales in AHP, too. [37, 38, 39].

Table 29. The 1 to 9 scale by Saaty (1977)

Intensity of importance	Definition
1	Equal importance
2	Weak
3	Moderate importance
4	Moderate plus
5	Strong importance
6	Strong plus
7	Very strong or demonstrated importance
8	Very, very strong
9	Extreme importance

Step 2. Consistency check

The maximum eigenvalue of the pairwise comparison of the criteria, λ_{max}, is equal to n if and only the matrix is consistent, in other words λ_{max} is greater than n if the matrix is not consistent. The function that describes the above is proposed by Saaty [40] and it proposes the Consistency Index (CI) as

$$CI(X) = \frac{\lambda_{\max} - 1}{n - 1}$$

However, the Consistency Index is not fully fair in comparing matrices with different orders and a rescale needs to be done, because the expected value of the Consistency Index for a random matrix with size n+1 is on average greater than the expected value of the Consistency Index for a random matrix with size n. So, the rescaled version of the Consistency Index, the Consistency Ratio (CR), is defined and is calculated by:

$$CR(X) = \frac{\mathrm{CI(X)}}{\mathrm{RI_n}}$$

The RI_n expresses the average Consistency Index which is obtained from a large data set of matrices with size n and they are randomly generated. Matrices with CR < 0.10 are acceptable, but matrices with CR > 0.10 are inconsistent [27].

Step 3. Priority vector of criteria

For the priority vector of criteria w = w_1, w_2,…, w_n there are several methods for its calculation. Here, three such methods will be explained and used in the case study.

a. Eigenvector method

Saaty [28]has proposed the most well-known method for the calculation of the priority vector, which is the principal eigenvector of *X*. In general, we have the formulation of the type *Xw = nw*, which implies that *n* and *w* are an eigenvalue and an eigenvector of *X*, respectively.

$$Xw = \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & \cdots & \cdots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} nw_1 \\ nw_2 \\ \vdots \\ nw_n \end{bmatrix} = nw$$

Vector w can be calculated for a pairwise comparison matrix X:

$$\begin{cases} Xw = \lambda_{max} \\ w^T[1, 1, \ldots, 1]^T = 1 \end{cases}$$

b. The normalized column sum method

The priority vector is expressed as the sum of all the elements in a row, which is further divided by the sum of the elements of the matrix *X* [27].

$$w_i = \frac{\sum_{j=1}^{n} x_{ij}}{\sum_{n=1}^{n} \sum_{j=1}^{n} x_{ij}}$$

c. Geometric mean method

Crawford and Williams [41] have proposed the geometric mean method, which calculates the priority vector through the division of the elements on a row by the normalization term, thus the sum of *w* is equal to 1.

$$w_i = \frac{\left(\prod_{j=1}^{n} x_{ij}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} x_{ij}\right)^{\frac{1}{n}}}$$

Step 4. Pairwise comparison matrices of the alternatives

The fourth step is similar to the first step, as a pairwise comparison matrix of the alternatives is created through the preferences of the decision-maker. The matrix is of size *mxm*, where m is the number of alternatives.

Step 5. Consistency check on the pairwise comparison matrices of the alternatives

The fifth step is similar to the second step, as a consistency check takes place on all pairwise comparison matrices of the alternatives, but here the number of criteria is replaced by the number of the alternatives.

Step 6. Compute the local priority vectors

The sixth step is similar to the third step, as the local priority vectors of all the alternatives is calculated and again the number of criteria is replaced by the number of the alternatives.

Step 7. Aggregate the local priorities – Rank the alternatives

In the last step, the global alternative priorities are calculated through the combination of the priority criteria and the local alternative priorities [27].

4.3.4 Example- Case study

A similar example that was used in the PROMETHEE method section will be used to further illustrate the steps of AHP. However, there are some important differences. The decision-maker does not use numerical evaluations, like those in the PROMETHEE method. Eventually, he/she will form the pairwise comparison matrices of the criteria and alternatives through his/her personal (and thus subjective) opinion about the preferences among the criteria. Therefore, there are again four criteria ('Capital Requirements', 'Available Staff', 'Delivery Distance' and 'Environments impact') and three alternatives (Option 1, Option 2 and Option 3).
Hypothetically, the following comparisons apply (table 31):

- Capital Requirements are very strong more important than the Available Staff (x_{12} = 7, x_{21} = 1/7).
- Capital Requirements are weakly important than the Environmental Impact (x_{14} = 2, x_{41} = 1/2).
- Delivery Distance is strong plus more important than the Available Staff

($x_{32} = 6$, $x_{23} = 1/6$).

- Environmental Impact is equally important asthe Delivery Distance ($x_{43} = x_{34} = 1$).
- Environmental Impact is weakly important than the Available Staff ($x_{42}= 2$, $x_{24} = 1/2$).
- Capital Requirements is weakly important than the Delivery Distance ($x_{13}= 2$, $x_{31} = 1/2$).

Table 30. Compare pairwise the importance of the criteria as regard to the goal

	Capital Requirements	Available Staff	Delivery Distance	Environmental impact
Capital Requirements	1	1/7	2	2
Available Staff	7	1	1/6	1/2
Delivery Distance	1/2	6	1	1
Environmental impact	1/2	2	1	1

4.3.4.1 Eigenvector method

In Table 32, we have the values for every criterion, using the eigenvector method. In detail, we have for the 'Capital Requirements' criterion:

$1 * 1 + \frac{1}{7} * 7 + 2 * \frac{1}{2} + 2 * \frac{1}{2} = 4,$

$7 * 1 + 1 * 7 + \frac{1}{6} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = 14.333$

and so on. Then we have the sum of each row and the eigenvector:

$32.381/181.190 = 0.179,$

$48.333/181.190 = 0.267$

and so on.

Table 31. Square the comparison matrix, sum the rows, and normalize

	Capital Requirements	Available Staff	Delivery Distance	Environmental impact	Sum of rows	Eigenvector
Capital Requirements	4.000	16.286	6.024	6.071	32.381	0.179
Available Staff	14.333	4.000	14.833	15.167	48.333	0.267
Delivery Distance	43.500	14.071	4.000	6.000	67.571	0.373
Environmental impact	15.500	10.071	3.333	4.000	32.905	0.182
				SUM	**181.190**	**1.000**

Hypothetically again, in the following tables33, 35, 37, 39 we have the comparison matrices for our three alternatives to every of the four criteria. In tables34, 36, 38 and 40 we have the local priorities respectively for every alternative to every criterion.

Table 32. Compare pairwise the alternatives ('Capital Requirements')

	Option 1	Option 2	Option 3
Option 1	1	6	3
Option 2	1/6	1	1/2
Option 3	1/3	2	1

Table 33. Calculate the local priorities with the eigenvector method ('Capital Requirements')

	Option 1	Option 2	Option 3	Sum of rows	local priorities
Option 1	3	18	9	30.000	0.667
Option 2	1/2	3	1 1/5	5.000	0.111
Option 3	1	6	3	10.000	0.222
			SUM	45.000	1.000

For instance, in table 34 we have for cell (Option 1 – Option 1):

$$1 * 1 + \frac{1}{6} * 6 + 2 * \frac{1}{2} + \frac{1}{2} * 2 = 3$$

and so on. Then we have the sum of the rows and the local priorities

$$30.000/45.000 = 0.667$$

and so on.

Table 34. Compare pairwise the alternatives ('Investment costs')

	Option 1	Option 2	Option 3
Option 1	1	4	2
Option 2	1/4	1	2
Option 3	1/2	1/2	1

Table 35. Calculate the local priorities with the eigenvector method ('Investment costs')

	Option 1	Option 2	Option 3	Sum of the rows	local priorities
Option 1	3	9	12	24.000	0.598
Option 2	1 1/2	3	4 1/2	9.000	0.224
Option 3	1 1/8	3	3	7.125	0.178
			SUM	40.125	1.000

Table 36. Compare pairwise the alternatives ('Delivery Distance')

	Option 1	Option 2	Option 3
Option 1	1	5	1/2
Option 2	1/5	1	1/3
Option 3	2	3	1

Table 37. Calculate the local priorities with the eigenvector method ('Delivery Distance')

	Option 1	Option 2	Option 3	Sum of rows	local priorities
Option 1	3	11 1/2	2 2/3	17.167	0.376
Option 2	1	3	3/4	4.833	0.106
Option 3	4 3/5	16	3	23.600	0.518
			SUM	45.600	1.000

Table 38. Compare pairwise the alternatives ('Environmental Impact')

	Option 1	Option 2	Option 3
Option 1	1	1/2	1/2
Option 2	2	1	4
Option 3	2	1/4	1

Table 39. Calculate the local priorities with the eigenvector method ('Environmental Impact')

	Option 1	Option 2	Option 3	Sum of rows	local priorities
Option 1	3	1 1/8	3	7.125	0.178
Option 2	12	3	9	24.000	0.598
Option 3	4 1/2	1 1/2	3	9.000	0.224
			SUM	40.125	1.000

The final stage is table 41, where we synthesize all the above and we calculate the global priorities of the alternative. As we can see, the Option 1 is first, the Option 2 is second and the Option 3 is the last one. As a result, the Option 1 is the best solution.

Table 40. Global priority of the alternatives (Eigenvector)

	Capital Requirements	Available Staff	Delivery Distance	Environmental impact	Criteria priorities	Global priority
Option 1	0.667	0.598	0.376	0.178	0.179	0.451
Option 2	0.111	0.224	0.106	0.598	0.267	0.228
Option 3	0.222	0.178	0.518	0.224	0.373	0.321
					0.182	1.000

We have the local priorities for each criterion, the eigenvector as the criteria priorities and finally the global priority:

$0.667 * 0.179 + 0.598 * 0.267 + 0.376 * 0.373 + 0.178 * 0.182 = 0.451$

and so on.

4.3.4.2 The normalized column sum method

Let us use the normalized column sum method and analytically to see if there are differences at every step of the methodology, and at the final results. Here, we are going to use the sum of every row, dividing it by the total sum, in order to find every criterion priority.

Table 41. Square the comparison matrix, sum the rows, and normalize

	Capital Requirements	Available Staff	Delivery Distance	Environmental impact	Sum of the rows	Criteria priorities
Capital Requirements	1	1/7	2	2	5.143	0.192
Available Staff	7	1	1/6	1/2	8.667	0.323
Delivery Distance	1/2	6	1	1	8.500	0.317
Environmental impact	1/2	2	1	1	4.500	0.168
				SUM	26.810	1.000

Thus, we have the sum of the rows and the criteria priorities: 5.143/26.81=0.192 and so on. In tables 43, 44, 45 and 46 we have the comparison matrices for all the alternatives as regards to every criterion.

Table 42. Local priority of the alternatives ('Capital Requirements')

	Option 1	Option 2	Option 3	Sum of the rows	local priorities
Option 1	1	6	3	10.000	0.667
Option 2	1/6	1	1/2	1.667	0.111
Option 3	1/3	2	1	3.333	0.222
			SUM	15.000	1.000

Table 43. Local priority of the alternatives '(Investment costs')

	Option 1	Option 2	Option 3	Sum of the rows	local priorities
Option 1	1	4	2	7.000	0.571
Option 2	1/4	1	2	3.250	0.265
Option 3	1/2	1/2	1	2.000	0.163
			SUM	12.250	1.000

Table 44. Local priority of the alternatives ('Delivery Distance')

	Option 1	Option 2	Option 3	Sum of the rows	local priorities
Option 1	1	5	1/2	6.500	0.463
Option 2	1/5	1	1/3	1.533	0.109
Option 3	2	3	1	6.000	0.428
			SUM	14.033	1.000

Table 45. Local priority of the alternatives ('Environmental Impact')

	Option 1	Option 2	Option 3	Sum of the rows	local priorities
Option 1	1	1/2	1/2	2.000	0.163
Option 2	2	1	4	7.000	0.571
Option 3	2	1/4	1	3.250	0.265
			SUM	12.250	1.000

Table 46. Global priority of the alternatives (normalized column sum)

	Capital Requirements	Available Staff	Delivery Distance	Environmental Impact	Criteria priorities	Global priority
Option 1	0.667	0.571	0.463	0.163	0.192	0.487
Option 2	0.111	0.265	0.109	0.571	0.323	0.238
Option 3	0.222	0.163	0.428	0.265	0.317	0.275
					0.168	1.000

For the global priority, we have:

0.667*0.192 + 0.323*0.238 + 0.463*0.317 + 0.160*0.168 = 0.487

and so on (Table 47). There are some minor differences to the global priorities considering the eigenvector method and the normalized column sum method. However, the order of the alternatives has not changed.

4.3.4.3 Geometric mean method

The final version is the geometric mean method for our example. Following the same procedure as before, we have the following tables. As the reader can see, here we use the geometric mean, which we calculate for each row by multiplying the values of each row and raising the result to 0.25.

Table 47. Square the comparison matrix, sum the rows, and normalize

	Capital Requirements	Available Staff	Delivery Distance	Environmental Impact	Geometric mean	Criteria priorities
Capital Requirements	1	1/7	2	2	0.87	0.214
Available Staff	7	1	1/6	1/2	0.87	0.215
Delivery Distance	1/2	6	1	1	1.32	0.324
Environmental Impact	1/2	2	1	1	1.00	0.246
				SUM	4.06	1.000

Thus, for the geometric mean, we have: $(1*(1/7)*2*2)^{0.25}$= 0.87 (Geometric mean for Capital Requirements) and so on. For the criteria priorities, we have: 0.87/4.06=0.214 (Criteria priorities for Capital Requirements) and so on.

Table 48. Local priority of the alternatives ('Capital Requirements')

	Option 1	Option 2	Option 3	Geometric mean	local priorities
Option 1	1	6	3	2.060	0.588
Option 2	1/6	1	1/2	0.537	0.153
Option 3	1/3	2	1	0.904	0.258
			SUM	3.501	1.000

Table 49. Local priority of the alternatives ('Investment costs')

	Option 1	Option 2	Option 3	Geometric mean	local priorities
Option 1	1	4	2	1.682	0.521
Option 2	1/4	1	2	0.841	0.260
Option 3	1/2	1/2	1	0.707	0.219
			SUM	3.230	1.000

Table 50. Local priority of the alternatives ('Delivery Distance')

	Option 1	Option 2	Option 3	Geometric mean	local priorities
Option 1	1	5	1/2	1.257	0.378
Option 2	1/5	1	1/3	0.508	0.153
Option 3	2	3	1	1.565	0.470
			SUM	3.331	1.000

Table 51. Local priority of the alternatives ('Environmental Impact')

	Option 1	Option 2	Option 3	Geometric mean	local priorities
Option 1	1	1/2	1/2	0.707	0.219
Option 2	2	1	4	1.682	0.521
Option 3	2	1/4	1	0.841	0.260
			SUM	3.230	1.000

Table 52. Global priority of the alternatives ('geometric mean')

	Capital Requirements	Available Staff	Delivery Distance	Environmental Impact	Criteria priorities	Global priority
Option 1	0.588	0.521	0.378	0.219	0.214	0.414
Option 2	0.153	0.260	0.153	0.521	0.215	0.267
Option 3	0.258	0.219	0.470	0.260	0.324	0.319
					0.246	1.000

Finally, here we have:

0.588*0.214 + 0.521*0.215 + 0.378*0.324 + 0.219*0.246 = 0.414

and so on. Comparing the final ranking with the results of the other two methods, there areagain some differences to the numbers of global priorities, but the order of the alternatives still remains the same.

4.3.5 Conclusions

As the reader can observe, there are some differences to the two methods described above, but despite their differences the order of the alternatives in the example has not changed. Perhaps, in another example, with more criteria and more alternatives, the use of each of the three methods could lead to quite different results, changing the order of the final ranking. However, here no big differences were observed. As a result, through the different variations the reader can easily understand why AHP is considered one of the most famous and versatile methods.

4.3.6 AHP Exercise

The local authorities of a region want to choosethe best form of transportation in order to make a new investment to the city. The available options are Metro, Trolley, and Tram. The local authorities are taking into account the following criteria:

- investment cost,
- transit time,
- accessibility,
- power requirements,
- social impact and
- environmental impact

For the above criteria, we have the following expressions:

- The Investment Cost is extreme more important than the Accessibility
  (x_{13} = 9, x_{31} = 1/9).
- The Power Requirements are very strong more important than the Transit Time
  (x_{42} = 7, x_{24} = 1/7).
- The Accessibility is equal important as the Social Impact
  (x_{35} = x_{53} = 1).
- The Environmental Impact is strong more important than the Social Impact
  (x_{65}= 5, x_{56} = 1/5).
- The Transit Time is weak important than the Environmental Impact
  (x_{26}= 2, x_{62} = 1/2).
- The Investment Cost is strong more important than the Transit Time
  (x_{12} = 5, x_{21} = 1/5).
- The Power Requirements are moderate more important than the Investment Cost
  (x_{41} = 3, x_{14} = 1/3).
- The Environmental Impact is strong more important than the Power Requirements

(x_{64} =5, x_{46} = 1/5).

- The Environmental Impact is strong more important than the Investment Cost (x_{61} = 5, x_{16} = 1/5).
- The Accessibility is equal important as the Transit Time (x_{32} = x_{23} = 1).
- The Transit Time is moderate more important than the Social Impact (x_{25} = 3, x_{52} = 1/3).
- The Power Requirements are strong more important than the Accessibility (x_{43} = 5, x_{34} = 1/5).
- The Investment Cost is strong more important than the Social Impact (x_{15} = 5, x_{51} = 1/5).
- The Environmental Impact is extreme more important than the Accessibility (x_{63} = 9, x_{36} = 1/9).
- The Power Requirements are moderate more important than the Social Impact (x_{45} = 3, x_{54} = 1/3).

We encourage the reader to solve this problem with the three methods that we have explained on the above sections (the eigenvector method, the normalized column sum method, and the geometric mean method) in order to understand completely the procedure of every method.

Table 53. Compare pairwise the importance of the criteria as regard to the goal

	Investment Cost	**Transit Time**	**Accessibility**	**Power Requirements**	**Social Impact**	**Environmental Impact**
Investment Cost	1	5	9	1/3	5	1/5
Transit Time	1/5	1	1	1/7	3	2
Accessibility	1/9	1	1	1/5	1	1/9
Power Requirements	3	7	5	1	3	1/5
Social Impact	1/5	1/3	1	1/3	1	1/5
Environmental impact	5	1/2	9	5	5	1

Multiple-choice questions

AHP stands for

- Analytical Hierarchy Program?
- Analytic Hierarchy Process?
- Analytical Hierarchy Programming?
- Analytic Hierarchy Partitioning?

The main purpose of AHP is to

- prioritize alternatives based on criteria and constraints?
- assign goals to alternatives?
- prioritize alternatives based on criteria?
- assign criteria to actions?

The typical Saaty scale is

- A 1–7 scale?
- A 1–3 scale?
- A 1–9 scale?
- A 1–17 scale?

Here there are some useful links explaining the AHP method with different exercises.

https://www.youtube.com/watch?v=J4T70o8gjlk
https://www.youtube.com/watch?v=3yTBcDP_JN8
https://www.youtube.com/watch?v=zsld4TQacBU

Test

Please press the following link to test your knowledge of the material covered in the book:
https://drive.google.com/file/d/1P5oK093I8b8QSL-YvpAMghU7hXDwUfsM/view?usp=sharing

1. Press the link. It opens a black page where it provides different options with what to open the file. Press "Google Collaboratory". Note that you must be connected to an active Google account in order to access the page.
2. It opens a page on the browser. There are three titles displayed: Problem Solving and Linear Programming, Data Envelopment Analysis, Multiple Criteria Decision Aid
3. Starting from Problem solving and Linear Programming, press the Run Cell/ Play button (or press Ctrl+Enter) below the title.
4. Answer the multiple choice questions
5. Repeat for Data Envelopment Analysis and Multiple Criteria Decision Aid

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List of Links

1. History of Data Envelopment Analysis https://www.youtube.com/watch?v=fxAASxxCeFc
2. Introduction to basics of DEA https://youtu.be/qE2mzqKTI58
3. Economies of Scale https://youtu.be/JdCgu1sOPDo
4. History of worker placement https://www.youtube.com/watch?v=4E6al0Lx80M
5. PROMETHEE https://www.youtube.com/watch?v=qQJO2UYJRc8
6. AHP https://www.youtube.com/watch?v=J4T70o8gjlk
7. AHP https://www.youtube.com/watch?v=3yTBcDP_JN8
8. AHP https://www.youtube.com/watch?v=zsld4TQacBU